Variations of Pseudoconvex Domains over C^n

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Various function-theoretic quantities can be associated to a domain D over the complex plane C. In [13]–[16] we studied how these quantities vary when D varies over C. An important quantity among them was the *Robin constant* which is defined as follows: Let D be an unramified covering domain over C whose boundary, ∂D, consists of smooth curves. For a fixed point ζ in D, the domain D carries the Green function $g(z)$ for Laplace's equation $\Delta g = 4\partial^2 g/\partial z\partial\bar{z} = 0$ with pole at ζ. The function g is uniquely determined by the following three conditions: $\Delta g = 0$ in D except at ζ, g is continuous up to ∂D and $g = 0$ on ∂D, and g differs from $\log(1/|z-\zeta|)$ by a harmonic function in a neighborhood of ζ. We put

$$\lambda = \lim_{z\to\zeta}\left(g(z) - \log\frac{1}{|z-\zeta|}\right).$$

Following Faber [3], we call λ the Robin constant for (D, ζ). Now we vary the domain D over C for t in the disk B: $|t| < \rho$; that is, we have a variation $t \to D(t)$ $(t \in B)$ with the following properties: $D(0) = D$, each $D(t)$ $(t \in B)$ is an unramified covering domain over C bounded by the smooth curves forming $\partial D(t)$, and each $D(t)$ contains the point ζ. We then have the Robin constant $\lambda(t)$ for $(D(t), \zeta)$. $\lambda(t)$ defines a real-valued function on B. In [15] we obtained the following.

THEOREM I. *If the set* $\mathbf{D} = \{(t,z) | t \in B \text{ and } z \in D(t)\}$ *is a pseudoconvex domain over the product space* $B \times C$, *then* $\lambda(t)$ *is a superharmonic function on* B.

The definition of a pseudoconvex domain over $B \times C$ is given in Oka [8, p. 101]. This theorem was motivated by Nishino's beautiful work on value distribution of entire functions of two complex variables (see his survey [7]). Theorem I has been recently applied to the theory of functions by Suzuki [10] and Fujita [4] and also to other areas by Wermer [12], Kaneko [6], and Suzuki [11].

In this paper we study the case when $D(t)$ varies over the complex n-dimensional Euclidean space C^n, where $n \geq 2$. Let D be an unramified covering domain D over C^n bounded by smooth surfaces ∂D. Fix $\zeta \in D$. Then

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