Zeros of the Successive Derivatives of Hadamard Gap Series in the Unit Disk

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1. Introduction

Pólya [8] defined the *final set* of an analytic function to be the set of all z in the complex plane such that every neighborhood of z contains zeros of infinitely many derivatives of f. He and others [1, 2, 3, 5, 6, 7, 8] have determined the final sets of various entire and meromorphic functions. Here we examine the final sets of Hadamard gap series with radius of convergence one (and hence with natural boundary $\{|z|=1\}$ [10, p. 223]).

We consider power series of the form

$$f(z)=\sum_{k=1}^{\infty} c_k z^{n_k}, \tag{1.1}$$

satisfying $\lambda>1$, where

$$\lambda=\liminf_{k\to\infty} n_{k+1}/n_k. \tag{1.2}$$

We define

$$H(\lambda)=\begin{cases}(\lambda-1)\lambda^{\lambda/(1-\lambda)} & \text{when } 1<\lambda<\infty,\\ 1 & \text{when } \lambda=\infty.\end{cases} \tag{1.3}$$

THEOREM 1. *Let f, λ, and $H(\lambda)$ be defined as above, and suppose that*

$$|c_k|^{1/n_k}\to 1 \quad \textit{as } k\to\infty. \tag{1.4}$$

(a) *If $1<\lambda<\infty$, then the final set of f is*

$$\{0\}\cup\{H(\lambda)\le|z|\le 1\}. \tag{1.5}$$

(b) *If $\lambda=\infty$, then the final set is contained in* (1.5). *If $\lambda=\infty$, and if* $\limsup n_k^B|c_k|>0$ *for some $B\ge 0$, then the final set is* (1.5).

In Section 4 some functions are constructed which satisfy (1.4), and for which $\lambda=\infty$, but for which the final set is $\{0\}$.

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