The Basic Geometry of the Manifold of Riemannian Metrics and of its Quotient by the Diffeomorphism Group

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Introduction

Let M be a compact, oriented, smooth n-manifold and consider the collection $\mathrm{Met}(M)$ of all Riemannian metrics on M. Although $\mathrm{Met}(M)$ is a contractible open cone inside the space $\Gamma(S^2T^*M)$ of symmetric rank-2 tensor fields, its natural metric (described later) is nonconstant and yields interesting geometry. Furthermore, the group $\mathrm{Diff}^+(M)$ of orientation-preserving diffeomorphisms acts isometrically (by pullback) on $\mathrm{Met}(M)$, and is free on the subset $\mathrm{Met}'(M)$ of metrics which admit no nontrivial isometries. Hence there is an induced metric on the quotient $\mathrm{Met}'(M)/\mathrm{Diff}^+(M)$. In this paper we derive formulas for the curvature and geodesics of $\mathrm{Met}(M)$ and of $\mathrm{Met}'(M)/\mathrm{Diff}^+(M)$.

The metric on $\mathrm{Met}(M)$ is an example of an "L^2 metric" on a mapping space. More generally, suppose M is a compact (finite-dimensional) manifold endowed with a measure μ, and let N be a Riemannian manifold with metric g. Then the space of (smooth) maps $\mathrm{Map}(M,N)$ inherits an L^2 metric as follows. A tangent vector at $\phi\in\mathrm{Map}(M,N)$ is a cross-section of the pulled-back tangent bundle $\phi^*TN\to M$, and the inner product of two tangent vectors A and B is

$$\langle A,B\rangle=\int_M g(A(x),B(x))\,\mu(x).$$

For this metric one easily calculates that the curvature $R(X,Y)Z$ is, pointwise, simply the curvature of N; it does not depend on the measure μ. Moreover, a geodesic in the mapping space corresponds to a family of geodesics in N. We discuss these matters in the appendix.

Although $\mathrm{Met}(M)$ is not, strictly speaking, a space of maps of the type above, it is the space of sections of a fiber bundle, and similar principles

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