The Structure of the Space of Co-Adjoint Orbits of a Completely Solvable Lie Group

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0. Introduction

Let G be a connected, simply connected, completely solvable Lie group with Lie algebra g. For G nilpotent, Pukanszky in [6] shows that there is an $\mathrm{Ad}^*(G)$-invariant Zariski open subset Ω of g^* in which all $\mathrm{Ad}^*(G)$-orbits have the same dimension and in which there is an algebraic subset Σ which is a cross-section for the orbits. Moreover, there is a subspace V of g^* and a computable, rational, nonsingular map $\Theta: \Sigma \times V \to \Omega$ such that, for each $\mathfrak{l} \in \Sigma$, $\Theta(\mathfrak{l}, \cdot)$ is a polynomial map whose graph is the orbit of $\mathfrak{l}$. In fact, Pukanszky's technique yields a layering of g^* by a collection of algebraic subsets $\{\Omega_j\}$ having a natural total ordering such that the maximal subset is Ω and such that in each Ω_j one can construct objects Σ_j, V_j, and Θ_j as described above. In this way a semi-algebraic cross-section for all the $\mathrm{Ad}^*(G)$-orbits is obtained. It should be emphasized that these constructions are quite explicit and depend only on the choice of a Jordan–Holder basis for g. The ordering of the layers and the computability of the cross-section in each layer makes this result particularly useful (see, e.g., [1]). For solvable groups, the layering $\{\Omega_j\}$ of g^* has itself been useful, but the space of co-adjoint orbits in each layer is more complex. For a given layer Ω, one cannot expect to obtain objects analogous to Σ, V, and Θ above. In this paper we show that, for G completely solvable, there is a refinement of the layering $\{\Omega_j\}$ such that in each of the refined layers one can obtain computable objects analogous to Σ, V, and Θ above. This refined layering also has a nice ordering, and the layers are algebraic sets. More specifically, we prove the following.

THEOREM. *Let G be a connected, simply connected, completely solvable Lie group with Lie algebra g, and let $g = g_n \supset g_{n-1} \supset \cdots \supset g_0 = (0)$ be a Jordan–Holder sequence of ideals in g. Choose a basis $X_1, X_2, \ldots, X_n$ for g such that $X_1, X_2, \ldots, X_j$ span g_j, and let $e_1, e_2, \ldots, e_n$ be the dual basis in g^*. Then there is a finite computable layering (i.e., partition) $\wp$ of g^* with the following properties:*

Received November 15, 1988.
The second author was partially supported by National Science Foundation grant #85-05771.
Michigan Math. J. 36 (1989).