A Fixed-Point Free Ergodic Flow on the 3-Sphere

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The 3-sphere S^3 supports a fixed-point free flow φ because the Euler characteristic of S^3 is zero; the same is true for any 3-manifold. However, it is not known what other properties may be imposed on φ. For example, the Seifert problem asks whether there exists a smooth flow on S^3 with neither fixed points nor periodic orbits. The problem is open. It is not even known whether S^3 supports a minimal flow—one whose only compact invariant sets are S^3 and $\emptyset$.

Ergodicity is a kind of measure-theoretic minimality—the only measurable invariant sets are of full measure or zero measure. In [6], Katok constructed examples of ergodic diffeomorphisms of surfaces. Here we point out how to "Birkhoff-suspend" them and induce flows on S^3 (or any lens space) that are smooth, ergodic respecting Lebesgue measure, *and have no fixed points*. Katok [7] has already given an example of such a flow on $\mathbf{P}^3$, but, being the geodesic flow of a semi-Finsler, it is somewhat difficult to picture. In contrast, the topology of our construction is fairly natural. It was used earlier by D. Fried (unpublished) to construct a smooth flow on S^3 for which the Lebesgue measure class is ergodic. (The measure which is invariant under Fried's flow might not be Lebesgue measure—it might not have a smooth positive density. However, its only measurable invariant sets have full or zero Lebesgue measure.) In our example, simultaneous smoothness of the flow and of the invariant ergodic measure requires some care.

It is not known if every 3-manifold supports a fixed-point free ergodic flow. This question may be related to the fact that Katok's construction takes place in the isotopy class of the identity. Whether ergodic diffeomorphisms exist in every isotopy class is not known.

Another issue is analyticity. It is not known if there is an analytic fixed point free ergodic flow on S^3. However, Gerber [5] has shown that analytic examples like Katok's do exist.

Also, if fixed points are permitted, the situation is entirely understood. Anosov and Katok [1] proved that ergodic flows exist on all manifolds M^m for $m \geq 3$, and Blohin [2] has constructed them on all surfaces except the sphere, projective plane, and Klein bottle, where they are impossible.

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