3-Dimensional Bordism

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In [10], C. Rourke gave an elementary proof of the following.

THEOREM 1 (Rohlin [9], Thom [11]). $\Omega_3^{SO}=0$ (*every closed oriented* 3-*manifold M is the oriented boundary of a compact oriented* 4-*manifold*).

Rourke's proof is by induction on Heegaard genus. He shows that if the genus of M is nonzero, then M is bordant to a disjoint union of manifolds of lower genus. The bordism is achieved by surgery on a carefully chosen link lying in the splitting surface of a Heegaard decomposition of M.

In this paper we show how to generalize Rourke's argument to give elementary proofs of the following two theorems.

THEOREM 2 (Rohlin [9], Thom [11]). $\Omega_3^{O}=0$ (*every closed* 3-*manifold is the boundary of a compact* 4-*manifold*).

THEOREM 3 (Milnor [6; 7]). $\Omega_3^{\mathrm{Spin}}=0$ (*every closed spin* 3-*manifold is the spin boundary of a compact spin* 4-*manifold*).

Compare Lickorish [5] for Theorem 2 and Kaplan [4] for Theorem 3.

The proof of Theorem 3 (or in particular Assertion 3, which is the chief contribution of this paper) gives an explicit construction of a family of simply-connected spin 4-manifolds with a given spin 3-manifold boundary. It is hoped that this family will be useful in the study of smooth closed simply-connected 4-manifolds and of invariants of 3-manifolds.

0. Preliminaries

We shall work in the smooth category. A *framing* $\mathbf{t}$ of a trivial vector bundle ϵ is a homotopy class of trivializations of ϵ, that is (up to homotopy) a list $t_1, \ldots, t_r$ of $r=\operatorname{rank}(\epsilon)$ linearly independent nonvanishing sections of ϵ. Write $\mathbf{t}=[t_1, \ldots, t_r]$.

Let M be an m-manifold. A framing of the restriction of the tangent bundle of M to a subset Z will be called a *tangential framing* of Z (in M). In particular, a *spin structure* on M is (for $m>2$) a tangential framing of

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