Asymptotic Behavior of Solutions of Oblique Derivative Boundary Value Problems

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1. Problems and Main Results

In the Euclidean space R^n let $L=(1/2)a^{ij}\partial_i\partial_j+b^i\partial_i$ be a uniformly elliptic operator and let $V=(V^1,\dots,V^n)$ be a vector field. Let q be a bounded nonnegative continuous function. Let D be a bounded domain and f a bounded measurable function on ∂D. Finally, let γ be a nontangential vector field on ∂D. Consider the solution u_f^ϵ of the boundary value problem $(\epsilon^2 L+V)u_f^\epsilon-qu_f^\epsilon=0$ on D, $\partial u_f^\epsilon/\partial\gamma=f$ on ∂D. In this paper, we study the asymptotic behavior of the solution u_f^ϵ as the parameter $\epsilon\to 0$. To guarantee the existence of a unique solution, we assume that q is not identically equal to zero on D. Under this condition, the solution tends to zero as ϵ goes to zero; the question is to find the appropriate exponential rate. This exponential rate depends on the behavior of the dynamical system $\dot{\phi}_s=V(\phi_s)$. We will discuss two typical cases: (1) the dynamical system has a unique equilibrium point in D; and (2) $V\equiv 0$. In the first case, we prove that $\lim_{\epsilon\to 0}\epsilon^2\log u_f^\epsilon$ exists and is equal to $-\inf_y I^+(x,y)$, where I^+ is the quasipotential function for the oblique derivative boundary value problem, and the infimum is taken over the essential support of the boundary value function f. In the second case, under the stronger condition that q is strictly positive on $\bar{D}$ and f is continuous, $\lim_{\epsilon\to 0}\epsilon\log u_f^\epsilon$ exists and the limit can also be explicitly identified.

The key to our discussion is a probabilistic representation of the solution u_f^ϵ. Let $\sigma=(\sigma^{ij})$ be a square root of the matrix $a=(a^{ij})$. Let $X=X^{x,\epsilon}$ be the solution of the stochastic differential equation with oblique reflection:

$$dX_t=\epsilon\sigma(X_t)dB_t+\epsilon^2 b(X_t)dt+V(X_t)dt-\gamma(X_t)\phi(dt),\quad X_0=x, \tag{1.1}$$

where B is a standard n-dimensional Brownian motion and ϕ is the boundary local time of the process X. Introduce the Feynman–Kac functional

$$e_q(t)=\exp\left\{-\int_0^t q(X_s)\,ds\right\}.$$

The solution u_f^ϵ can be represented explicitly as

Received July 18, 1988. Revision received December 24, 1988.
This research was supported in part by NSF grant 86-01977.
Michigan Math. J. 36 (1989).