Degree Bounds for the Division Problem in Polynomial Ideals

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1. Introduction

Let $f_1, \ldots, f_m$ be polynomials in n variables of degree at most D over an arbitrary field, and let I be the ideal they generate. We are interested in the following problem: If $P \in I$, what is the smallest integer δ such that we can write

$$P = f_1 g_1 + \cdots + f_m g_m \tag{1.1}$$

with $\deg g_j \leq \delta$? In general δ can be quite large; for example, [16] constructs examples for which δ is larger than D^{a^n}, where a is a positive constant (see also [2]). The doubly exponential estimate $\delta \leq (mD)^{2^n} + \deg P$ was given in 1926 by Hermann [11] (see the Appendix in [16]). For the case of the complex field, Berenstein and Yger [4] gave a better estimate for δ when the zero locus of I is zero-dimensional; another estimate for δ when $\{f_1, \ldots, f_m\}$ is a regular sequence is given in [3, Thm. 4.1]. In this paper we give sharp bounds for δ under the assumption that the zero locus of I is zero-dimensional at all finite and infinite points (Theorem 2). It remains an open problem whether this bound remains valid under the weaker hypothesis of [4]. Our method involves replacing the f_j by their homogenizations in $n+1$ variables and localizing in projective n-space. We first show (Theorem 1) that if I is homogeneous and of height $n-1$ then I is $(nD-n+1)$-regular; in particular, I is $(nD-n+1)$-saturated, which allows us to localize. (A result similar to Theorem 1 is given by Briançon [5, Prop. 4].)

In the special case $P=1$, in which (1.1) is called the *Bezout equation*, much better estimates can be found. Estimates for the Bezout equation have been given by Brownawell [6; 7], Masser and Wüstholz [15], Thompson [18], and recently by Caniglia, Galligo, and Heintz [8]. After the first version of this paper was completed, Kollár [12] obtained a sharp degree bound for the Nullstellensatz that yields the sharp result $\delta \leq D^{\min(m,n)} - D$ for the Bezout equation when $D \geq 3$.

Throughout this paper, we let K denote an algebraically closed field. We let $\mathbf{A}^n = \mathbf{A}^n_K$ and $\mathbf{P}^n = \mathbf{P}^n_K$ denote affine n-space and projective n-space (respectively) over K. By a *point* in $\mathbf{A}^n$ or $\mathbf{P}^n$ we mean a closed point. If V is an

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