A Maximum Principle for Sums of Subharmonic Functions, and the Convexity of Level Sets

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Introduction

We show in this paper that certain subharmonic functions satisfy a maximum principle on certain subvarieties of their domains of definition. Its statement (Theorem 1.4) is somewhat reminiscent of the fact that restrictions of plurisubharmonic functions in $\mathbf{C}^n$ to the zero-variety Z of some holomorphic function have no strict local maxima on Z. We discovered it in the course of proving the following convexity property of harmonic functions.

THEOREM I. *If* $n \geq 2$ *and*

(a) W *is a bounded convex open subset of* R^n,
(b) K *is a compact convex subset of* W,
(c) $\Omega = W \backslash K$,
(d) $u : \bar{\Omega} \to R$ *is continuous, harmonic in* Ω, *and* $u = 0$ *on* ∂K, $u = 1$ *on* ∂W,

then every level set of u *in* Ω *is a strictly convex hypersurface.*

Only after proving this did we realize that we were not the first to do so. In fact, there exist several proofs of this and of related results ([1], [2], [8], [9], [10]). However, our proof is different from these; it is quite elementary, and it gives some new *quantitative* information: Theorem 4.5 shows that all level sets of u are "at least as convex" as are ∂W and ∂K. (Similar quantitative information has been found quite recently by Dennis Stowe, again by entirely different methods.) We also hope that our maximum principle may have some further applications.

The term "strictly convex" refers to the Hessian of u. This is defined to be the quadratic form

$$H_p(u, \xi) = \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j}(p)\xi_i \xi_j,$$

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