A New Proof that a Mapping Is Regular If and Only If It Is Almost Periodic

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I. Introduction and Definitions

Throughout this paper, X will denote a compact metric space with metric d. The uniform metric on the space of self mappings on X will be denoted by ρ.

A homeomorphism f of a compact metric space onto itself is *regular* if and only if the family of iterates $\{\dots, f^{-2}, f^{-1}, f^0, f^1, f^2, \dots\}$ is an equicontinuous family. A mapping f of a compact metric space onto itself is *almost periodic* if and only if the following is true: If $\epsilon > 0$, then there exists a positive integer N such that every block of N consecutive positive integers contains an integer n such that $d(x, f^n(x)) < \epsilon$ for all $x \in X$ $(\rho(f^n, \mathrm{id}) < \epsilon)$. In case f is a homeomorphism, then the negative iterates are included as well. See Theorem F below.

Motivated by a desire to understand the mechanics of regular mappings, we give a self-contained proof of the theorem in the title of this paper. For an older proof see [1]. It is hoped that the present-day interest in topological dynamics will be served by a fresh proof of this useful old theorem. A clue to the argument is contained in the appendix to [1, p. 146] and is incorporated here into Lemma 1. We will also make use of the following theorem, one proof of which can be found in [3, Lemma 2.2]. But in the spirit of self-containment, we give an outline of the proof here.

THEOREM F. *If f is a mapping of a compact metric space onto itself whose positive iterates form an equicontinuous family (positively regular) then f is a regular homeomorphism (which will henceforth be referred to simply as a regular mapping).*

Outline of proof. We begin by assuming the following proposition, the proof of which is straightforward:

(∗) If the positive iterates of f form an equicontinuous family then either f is a homeomorphism or there exists $\delta > 0$ such that $\rho(f^i, \mathrm{id}) \geq \delta$ for $i = 1, 2, 3, \dots$.

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