CYCLIC VECTORS FOR MULTIPLICATION OPERATORS

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A vector f in a (complex) linear topological space is said to be a *cyclic* vector for a continuous linear transformation T on E if the set $\mathcal{P}(T)f$ is dense in E. Here $\mathcal{P}(T)=\{p(T): p\in\mathcal{P}\}$, and $\mathcal{P}$ denotes the set of polynomials. We prove two theorems about cyclic vectors in spaces of functions, first for measurable functions, then for analytic functions. The corollary to the first theorem generalizes a theorem of Bram about cyclic vectors of normal operators on Hilbert space.

If μ is a finite measure, let X denote the closed unit ball of $L^\infty(\mu)$ and let X_p denote X with the metric of $L^p(\mu)$. Recall that a residual subset of a complete metric space is a subset whose complement is a set of the first category.

THEOREM 1. *Let μ be a compactly supported Borel measure in the complex plane. If $0<p<\infty$, then:*

(a) *X_p is a complete metric space;*

(b) *the set of cyclic vectors in $L^p(\mu)$ for the operator of multiplication by z is a residual set; and*

(c) *the subset of cyclic vectors that lie in X is a residual subset of X_p.*

Proof. Choose $p\in(0,\infty)$ and keep it fixed throughout the proof.

(a) If $\{f_n\}\subset X$ and if $f_n\to f$ in $L^p(\mu)$, then f_n converges to f in measure, and a subsequence converges pointwise almost everywhere; thus $f\in X$ and so X_p is complete.

(b) Let $\{U_n\}$ be a countable basis for the open sets in $L^p(\mu)$. Let T denote the operator of multiplication by z. For each n, let V_n denote the set of vectors f for which there exists a polynomial p such that $p(T)f\in U_n$. Then V_n is an open set, and $\bigcap V_n$ is the set of cyclic vectors for T. Thus (b) will be proved if we show that V_n is dense in $L^p(\mu)$.

Let $K=\operatorname{supp}(\mu)$. If F is a closed set, if κ_F is its characteristic function, and if h is any function, then h_F denotes the function $\kappa_F h$. Finally, let $d(f,g)$ denote the distance from f to g in $L^p(\mu)$.

Claim 1. Let $\epsilon>0$ and $h\in L^p(\mu)$ be given. Then there exists $\delta>0$ such that if $F\subset K$ and $\mu(K\backslash F)<\delta$, then $d(h,h_F)<\epsilon$. Indeed, if $p\le 1$ then

$$d(h,h_F)=\int|h-h_F|^p\,d\mu=\int_{K\backslash F}|h|^p\,d\mu,$$

and the result follows by absolute continuity. The proof is similar when $p>1$.

Claim 2. If $\delta>0$ is given, then there exists a closed set F with empty interior and connected complement, such that $F\subset K$ and $\mu(K\backslash F)<\delta$. Indeed, let $\{I_n\}$ be an enumeration of the open subintervals with rational endpoints on the real axis,

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