SUBNORMAL TUPLES QUASI-SIMILAR TO THE SZEGÖ TUPLE

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In what follows, if $\mathcal{H}$ is a Hilbert space then $\mathcal{B}(\mathcal{H})$ denotes the set of bounded linear operators in $\mathcal{H}$. All the Hilbert spaces occurring below are separable and all the measures are compactly supported positive regular Borel measures on $\mathbf{C}^m$. Recall that if $S_1, \dots, S_m$ are m commuting elements in $\mathcal{B}(\mathcal{H})$, then the operator tuple $S = (S_1, \dots, S_m)$ is called *subnormal* on $\mathcal{H}$ if there exist a Hilbert space $\mathcal{H}' \supset \mathcal{H}$ and m commuting normal elements $N_1, \dots, N_m$ in $\mathcal{B}(\mathcal{H}')$ such that $N_j \mathcal{H} \subset \mathcal{H}$ and $N_j/\mathcal{H} = S_j$ for $1 \le j \le m$. If $H^2(\mathbf{B}^{2m})$ denotes the Hardy space of the open unit ball $\mathbf{B}^{2m}$ in $\mathbf{C}^m$ [i.e., $H^2(\mathbf{B}^{2m})$ is the completion of polynomials in $L^2(\sigma)$, σ being the surface area measure on the unit sphere $\mathbf{S}^{2m-1}$], and $M_{z_j}^{(\sigma)}$ denotes multiplication by z_j on $H^2(\mathbf{B}^{2m})$; then the multiplication tuple $M_z^{(\sigma)} = (M_{z_1}^{(\sigma)}, \dots, M_{z_m}^{(\sigma)})$, hereafter referred to as the Szegö tuple, is an example of a subnormal tuple. Moreover, $M_z^{(\sigma)}$ is cyclic. Recall that an operator tuple $S = (S_1, \dots, S_m)$ on $\mathcal{H}$ is called *cyclic* if there exists a vector u in $\mathcal{H}$ (called a *cyclic vector* for S) such that the smallest subspace of $\mathcal{H}$ containing u and invariant under $S_1, \dots, S_m$ is all of $\mathcal{H}$. The constant function 1 of course serves as a cyclic vector for $M_z^{(\sigma)}$. The following proposition is a well-known fact about cyclic subnormal tuples [3].

PROPOSITION 0. *Suppose $S = (S_1, \dots, S_m)$ is a subnormal tuple on $\mathcal{H}$ with a cyclic vector of norm one. Then there exists a probability measure μ with compact support in $\mathbf{C}^m$ and a unitary operator U from $\mathcal{H}$ onto $H^2(\mu)$ [$H^2(\mu)$ is the completion of polynomials in $L^2(\mu)$] such that $\mathsf{U}u = 1$ and $S_j = \mathsf{U}^* M_{z_j}^{(\mu)} \mathsf{U}$, $1 \le j \le m$; where $M_{z_j}^{(\mu)}$ is multiplication by z_j on $H^2(\mu)$.*

DEFINITION. Let $S = (S_1, \dots S_m)$ be a subnormal tuple on $\mathcal{H}$, and let $T = (T_1, \dots T_m)$ be a subnormal tuple on $\mathcal{K}$. We say that S is *quasi-similar to* T if there exist bounded linear operators $A: \mathcal{H} \to \mathcal{K}$ and $B: \mathcal{K} \to \mathcal{H}$ such that $\operatorname{Ker} A = \{0\}$, $\operatorname{Ker} B = \{0\}$, $\overline{\operatorname{Ran} A} = \mathcal{K}$, $\overline{\operatorname{Ran} B} = \mathcal{H}$, and $AS = TA$, $SB = BT$; that is, $AS_j = T_j A$ and $SB_j = BT_j$ for $1 \le j \le m$.

A function-theoretic characterization of subnormal tuples quasi-similar to the multiplication tuple on the Hardy space of the unit polydisc was obtained in [3]. In this note we observe that a similar characterization holds for subnormal tuples quasi-similar to the Szegö tuple. Our characterization allows us in particular to recapture a result in [2] that the Bergman tuple is not quasi-similar to the Szegö tuple. [If $A^2(\mathbf{B}^{2m})$ denotes the completion of polynomials in $L^2(V)$, V being the volumetric measure on the closed unit ball $\overline{\mathbf{B}^{2m}}$, then the multiplication tuple on $A^2(\mathbf{B}^{2m})$ is called the Bergman tuple.]

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