ANALYTIC CONTINUATION OF BIHOLOMORPHIC MAPS

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In this note we give another proof of the following result of Baouendi, Jacobowitz, and Treves [1].

THEOREM. *Let $\Omega_1, \Omega_2 \subset \mathbf{C}^n$ be domains with real analytic boundaries, and let $f: \Omega_1 \to \Omega_2$ be a biholomorphism which extends as a diffeomorphism $f: \bar{\Omega}_1 \to \bar{\Omega}_2$. If $p \in \partial\Omega_1$ and if there is no nontrivial complex variety in $\partial\Omega_2$ passing through $f(p)$, then f extends holomorphically to a neighborhood of p.*

More general results have been obtained by several authors; see Baouendi and Rothschild [2] and Diederich and Fornaess [4] and the references there. The proof given here applies in more general situations. It is evident, for instance, that the proof applies most naturally to the condition that $\partial\Omega_2$ have essentially finite type at p.

We let $\Gamma_f \subset \Omega_1 \times \Omega_2$ denote the graph of f. In what follows, we will show that there is a germ V of an n-dimensional variety in $\mathbf{C}^n \times \mathbf{C}^n$ containing $(p, f(p))$ and Γ_f. It will then follow from Lemma 1 of [3] that f extends holomorphically past p.

We may assume that $p=0$ and that $\partial\Omega_1 = \{\varphi(\zeta, \bar{\zeta}) = 0\}$ near 0, where $\varphi(\zeta, \bar{\eta})$ is analytic in ζ and $\varphi(\zeta, \bar{\eta}) = \overline{\varphi(\eta, \bar{\zeta})}$. We may assume also that $\varphi = \frac{1}{2}(\zeta_n + \bar{\eta}_n) + \cdots$, so that $\{\operatorname{Re} \zeta_n = 0\}$ is the tangent plane to $\partial\Omega_1$ at 0. Thus

$$E = \{\operatorname{Re} \zeta_1 = \cdots = \operatorname{Re} \zeta_{n-1} = 0\} \cap \Omega_1$$

is a totally real n-manifold, and the reflection about E is given by solving the complexification of the real defining equations: $\zeta_j + \bar{\zeta}_j^* = 0$, $1 \le j \le n-1$, and $\varphi(\zeta, \bar{\zeta}^*) = \frac{1}{2}(\zeta_n + \bar{\zeta}_n^*) + \cdots = 0$.

Thus the reflection about E is an antiholomorphic map of the form:

$$(\zeta_1^*, \ldots, \zeta_n^*) = -(\bar{\zeta}_1, \ldots, \bar{\zeta}_n) + \cdots.$$

We let Ω_1^* denote the image of Ω_1 under this reflection so that $E \subset \partial\Omega_1 \cap \partial\Omega_1^*$ and $T_0\, \partial\Omega_1^* = T_0\, \partial\Omega_1$, although the outward normals point in opposite directions at 0.

Let us start with $\tilde{X}_j = \partial_{z_j} - (\varphi_{z_j}/\varphi_{z_n})\partial_{z_n}$, $1 \le j \le n-1$, and $\tilde{X}^\alpha = \tilde{X}_1^{\alpha_1} \cdots \tilde{X}_{n-1}^{\alpha_{n-1}}$. We then define X_j and X^α by setting $X_j = \tilde{X}_j$ and $X^\alpha = \tilde{X}^\alpha$ on E, and extending them from E by making the coefficients holomorphic in a neighborhood of E. Thus X^α is tangential to $\partial\Omega_1$ at points of $\partial\Omega_1 \cap E$. Although $X^\alpha \ne (X_1)^{\alpha_1} \cdots (X_{n-1})^{\alpha_{n-1}}$, the highest-order parts of both operators are equal to ∂_z^α at 0.

Now let $f(0)=0$, and let $\psi(w, \bar{w})$ be a defining function for Ω_2. It follows that the (antiholomorphic) operators $\bar{X}^\alpha$ annihilate $\psi(f(z), \overline{f(z)})$ along E. By the chain rule, we obtain an expression of the form:

$$\bar{X}^\alpha \psi(f(z), \overline{f(z)}) = \sum \partial_{\bar{w}}^\gamma \psi P_\gamma \overline{(X^{\alpha_i} f_j(z))},$$

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