A STABLE SPLITTING FOR THE MAPPING CLASS GROUP

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Introduction. Let $\Gamma_{g,r}^{s}$ denote the mapping class group of a surface of genus g with r boundary components and s punctures. Because of their close connection with moduli spaces of algebraic curves, the homological and homotopy-theoretic properties of these groups are particularly interesting. Many of these properties are independent of the genus g, providing g is sufficiently large. It is therefore convenient to define a limit group $\Gamma = \varinjlim \Gamma_{g,1}^{0}$ (see §1 for details). By a theorem of Harer [7], the homology of Γ is the same as that of $\Gamma_{g,r}^{0}$ for any r, in degrees $i \ll g$. Moreover, Γ is a perfect group; hence we can form a simply connected space $B\Gamma^{+}$ with the same homology as Γ. The space $B\Gamma^{+}$ has a natural H-space structure and there is a natural H-map $B\Gamma^{+} \to B\mathrm{GL}(\mathbf{Z})^{+}$. Using this map, one can derive homotopy properties of $B\Gamma^{+}$ from those of $B\mathrm{GL}(\mathbf{Z})^{+}$.

In particular, Quillen ([12], [13]) showed that the maps $B\mathrm{GL}(\mathbf{Z})^{+} \to B\mathrm{GL}(\mathbf{F}_q)^{+}$ induced by reduction mod q split when localized at appropriate primes p; or, in other words, that $B\mathrm{GL}(\mathbf{Z})^{+}$ splits as a product of spaces

$$B\mathrm{GL}(\mathbf{Z})^{+} \simeq \mathrm{Im}\, J_{(1/2)} \times ?,$$

where

$$\mathrm{Im}\, J_{(1/2)} = \prod_{p \text{ odd}} B\mathrm{GL}(\mathbf{F}_q)^{+}_{(p)}$$

(cf. §2). In [3], Charney and Lee prove that the composite map

$$B\tau : B\Gamma^{+} \to B\mathrm{GL}(\mathbf{Z})^{+} \to \mathrm{Im}\, J_{(1/2)}$$

induces a split surjection on homology. It is natural to ask, therefore, whether this composite also splits on the space level. This question remains unanswered, but in this paper we prove that there is a *stable* splitting of $B\tau$ (Theorem 3.1) and hence a splitting of spaces (Corollary 3.2),

$$\Omega^{\infty}\Sigma^{\infty}B\Gamma \simeq \Omega^{\infty}\Sigma^{\infty}\,\mathrm{Im}\, J_{(1/2)} \times ?.$$

Analogous splitting theorems for $B\Gamma_{g,r}^{s}$ have been proved by the second author in [4] in some special cases. For example, a stable splitting of $B\Gamma_{0,0}^{4}$ is given in terms of the symmetric group on four letters and the Steinberg idempotent for $\mathrm{GL}_2(\mathbf{F}_2)$. In another example, it is shown that there exists a homomorphism θ: $\mathbf{Z}/5 \to \Gamma_{2,0}^{0}$ which induces an isomorphism on the 5-primary component of homology, and in [1] Benson uses these results to give a complete calculation of $H_*(\Gamma_{2,0}^{0})$.

The first two sections of this paper contain definitions and some well-known facts about mapping class groups and general linear groups. Section 3 contains

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