# THE NUMERICAL RADIUS OF A COMMUTING PRODUCT

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The numerical radius $w(T)=\sup\{|(Tx,x)|, x\in H, \|x\|=1\}$ is, apart from the norm and the spectral radius, one of the most important constants attached to a bounded operator $T$ in a complex Hilbert space $H$. There exists an extensive theory concerning the numerical radius and its relations to the norm.

In the present paper we investigate the following question:

(1) Is it true that $w(TS)\le w(T)\|S\|$ for all commuting operators $T$ and $S$ in a Hilbert space?

This question was probably first considered by Holbrook ([5], [6]) and further studied (usually in the more general context of operator radii $w\rho$ and $C\rho$ contractions) by a number of authors (see e.g. [1], [2], [3], [8]). For more about the history and motivations of the problem see [7].

Conjecture (1) looks very reasonable as there are positive results which indicate that the inequality might be true. The inequality is true if $T$ and $S$ are doubly commuting; that is, if $TS=ST$ and $TS^*=S^*T$ (Holbrook [5], Sz.-Nagy [8]). Also, if $S$ is an isometry then $w(TS)\le w(T)$ (Bouldin [2]). Finally, if $T$ and $S$ are arbitrary commuting operators then $w(TS)\le 1.169w(T)\|S\|$ (due to Crabb, communicated by Ando and Okubo [1]).

The aim of this paper, however, is to show that the conjecture is false in general. We exhibit an example of two operators $T, S$ in a 12-dimensional Hilbert space $H$ such that $TS=ST$, $\|S\|\le 1$, $w(T)\le 1$, and $w(TS)>1$. Using a result of Holbrook [7] it is even possible to assume that $S$ is a polynomial of $T$.

The Hilbert spaces considered in this paper are complex, but the constructed example works in real Hilbert space also.

We start with the following well-known lemma:

LEMMA. *Let $n$ be a positive integer, and let $a_{ij}$ $(i,j=1,\dots,n)$ be complex numbers such that the matrix $(a_{i,j})_{i,j=1}^n$ is positive definite. Then there exist a Hilbert space $H$ $(\dim H=n)$ and linearly independent elements $x_1,\dots,x_n\in H$ such that $(x_i,x_j)=a_{j,i}$ $(i,j=1,\dots,n)$.*

*Proof.* Let $H$ be an $n$-dimensional linear space with a basis $x_1,\dots,x_n$. Define the scalar product $(\cdot,\cdot)$ on $H$ by

$$\left(\sum_{i=1}^n \alpha_i x_i, \sum_{j=1}^n \beta_j x_j\right)=\sum_{i,j=1}^n \alpha_i\bar{\beta}_j a_{j,i} \quad (\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n\in\mathbf{C}).$$

Clearly, $H$ will become a Hilbert space, $\dim H=n$, and $(x_i,x_j)=a_{j,i}$ for $i,j=1,\dots,n$.

Received August 28, 1987. Revision received January 18, 1988.
Michigan Math. J. 35 (1988).