GENERAL ELEMENTS AND JOINT REDUCTIONS

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Introduction. Throughout this paper we will be concerned with a local ring (Q, m, k, d), where this implies that the local ring Q has maximal ideal m, residue field $k = Q/m$, and Krull dimension d. The general extension Q_g of Q will play an important role, and is defined as follows. We suppose that $X_1, X_2, \ldots$ is a countable set of indeterminates over Q. Then Q_g is the localization of the ring $Q[X_1, X_2, \ldots]$ at the prime ideal $m[X_1, X_2, \ldots]$. Q_g is a local Noetherian ring, the fact that it is Noetherian following from Proposition 1 of [1, Ch. 9]. It is a flat extension of Q and has maximal ideal $m_g = mQ_g$, its residue field is $k(X_1, X_2, \ldots)$ and it has Krull dimension d. It is also the union of the local rings Q_N, defined as the localization of $Q[X_1, \ldots, X_N]$ at $m[X_1, \ldots, X_N]$.

Now we come to the definition of general elements. Let $\underline{I} = (I_1, \ldots, I_s)$ be a set of ideals of Q, not necessarily distinct. We first define a standard independent set of general elements of $\underline{I}$ as follows. Let $a(i,1), \ldots, a(i, n_i)$ be a set of generators of I_i for $i = 1, \ldots, s$. Write $X(i,j)$ for X_h, where $h = n_1 + \cdots + n_{i-1} + j$ with $0 < j \le n_i$. Finally, let $x_i = \sum X(i,j) a(i,j)$, the sum being from $j = 1$ to n_i. Then we term the elements $x_1, \ldots, x_s$ a standard independent set of general elements of $\underline{I}$.

We now define an independent set of general elements of $\underline{I}$ to be a set of elements $x_1, \ldots, x_s$ of Q_g such that there exists an automorphism θ of Q_g, which fixes the elements of Q and the elements X_i for all sufficiently large i, such that the set of elements $\theta(x_i)$ is a standard set of general elements of $\underline{I}$. We shall prove below that this definition is independent of the choice of the sets of generators of the ideals I_i used in the definition of standard sets of independent general elements, by proving that any one set of independent general elements of $\underline{I}$ can be taken into any other such set by applying a suitable automorphism of Q_g of the type indicated. This implies that the ideal $X(\underline{I}) = (x_1, \ldots, x_s) \cap Q$ of Q is independent of the choice of the independent set of general elements $x_1, \ldots, x_s$ and that the Q-algebra $Q_g/(x_1, \ldots, x_s)$ is independent of the choice of $x_1, \ldots, x_s$ to within isomorphism as a Q-algebra.

Now we turn to the second term in our title, joint reductions. We recall that if I and $J \subseteq I$ are two ideals of a Noetherian ring then J is termed a reduction of I if $I^{r+1} = I^r J$ for some r (and hence all large r). Now suppose that $\underline{I} = (I_1, \ldots, I_d)$ is a set of d m-primary ideals of Q. Then we term a set of elements $y_1, \ldots, y_d$ of Q (with y_i in I_i) a joint reduction of $\underline{I}$ if, for some r,

$$(I_1 \cdots I_d)^{r+1} = \sum_{i=1}^{d} y_i (I_1 \cdots I_d)^r I_1 \cdots I_{i-1} I_{i+1} \cdots I_d.$$

In general, joint reductions need not exist, although they do if k is infinite. In particular, if $\underline{I}_g = (I_1 Q_g, \ldots, I_d Q_g)$ then any set of independent general elements

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