THE SINGULAR SET OF A NONLINEAR ELLIPTIC OPERATOR

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0. Introduction. The equation

$$\Delta u + \lambda u - u^3 = g \text{ on } \Omega, \quad u \mid \partial\Omega = 0,$$

where $\Omega \subset \mathbf{R}^n$ $(n \le 4)$ is a bounded domain, was studied in [9], and we continue that study here. Let H be the Sobolev space $W_0^{1,2}(\Omega)$, define

$$\langle A_\lambda(u), \varphi \rangle_H = \int_\Omega [\nabla u \nabla \varphi - \lambda u \varphi + u^3 \varphi]$$

for all $\varphi \in C_0^\infty(\Omega)$, and define $A: H \times \mathbf{R} \to H \times \mathbf{R}$ by $A(u, \lambda) = (A_\lambda(u), \lambda)$. The present paper investigates the singular set SA of this (real analytic) mapping.

Most results here are actually given for a more general nonlinear operator $A_\lambda: H \to H$ called abstract A_λ (0.1) defined on some Hilbert space, and the map defined in the previous paragraph is called standard A_λ (0.2).

Let $\lambda_j(u)$ be the jth eigenvalue of $DA_\lambda(u)$ (for standard A, of $\Delta v + \lambda v - 3u^2 v = 0$); then the singular set SA is the union of graphs of these eigenvalue functions $\lambda_j: H \to \mathbf{R}$ $(j = 1, 2, \dots)$, each is locally Lipschitzian (1.5), and wherever $\dim \ker DA_{\lambda_j}(\bar{u}) = 1$, the function λ_j at $\bar{u}$ is locally real analytic (1.8). In particular, for $\lambda < \lambda_2$ (the second eigenvalue of $-\Delta$ with null boundary conditions), SA is the graph of a real analytic function $\lambda_1: U \to \mathbf{R}$, where U is an open star-shaped neighborhood of 0 in H (1.9 and 2.4); moreover the function λ_1 has its only singular point at $u = 0$ (1.10), but it fails to satisfy the Morse lemma (2.10). For standard A the function λ_1 is real analytic for all $u \in H$ (1.9). Moreover, if $\partial\Omega$ is a compact C^∞ manifold, then there are (1.11) an open dense subset V_j of H such that $\lambda_j \mid V_j: V_j \to \mathbf{R}$ is real analytic $(j = 1, 2, \dots)$, and (1.12) an open dense subset W of the singular set SA such that, for every $(u, \lambda) \in W$, $\dim \ker DA_\lambda(u) = 1$.

If $\lambda < \lambda_{j+1}$ and $0 \ne u \in H$, then the ray $\{cu : c \ge 0\}$ meets SA_λ in at most j points (2.5). On the other hand, given any $\lambda \in \mathbf{R}$, there is a $0 \ne u \in H$ such that the line $\{cu : c \in \mathbf{R}\}$ is disjoint from SA_λ if $\lambda \ne \lambda_j$, and for $\lambda = \lambda_j$ they meet only in $(0, \lambda_j)$ $(j = 1, 2, \dots)$ (2.6). Thus, for any $\lambda > \lambda_1$ there is a $0 \ne u \in H$ such that $\lambda_1 \le \lambda_1(cu) < \lambda$ for all $c \in \mathbf{R}$ (2.7). If $A_\lambda(u) = 0$, $u \in SA_\lambda$, and $\lambda \le \lambda_k$, then (3.1)

$$(u, \lambda) \in \left(\bigcup_{i=1}^{k-2} \operatorname{graph} \lambda_i \right) \cup \{(0, \lambda_{k-1}), (0, \lambda_k)\}.$$

Our ultimate goal is to determine for each g and λ the number of (weak) solutions u of the given boundary value problem, and how this number changes as

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