CLOSED IDEALS IN CONVOLUTION ALGEBRAS AND THE LAPLACE TRANSFORM

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Introduction. Let $L^1[0,1]^n$ denote the Banach algebra of integrable functions on $[0,1]^n$ with restricted convolution as multiplication. It is easy to prove that all closed ideals in $L^1[0,1]$ are of the form:

$$M_\beta = \{f : \inf(\text{essential support}(f)) \geq \beta\} \quad \beta \in [0,1].$$

(See Section 2.) Thus, a function f in $L^1[0,1]$ generates a dense ideal in $L^1[0,1]$ if and only if zero is in the essential support of f. We demonstrate that this is not true for $n>1$ and then describe a relationship between the closed ideals in $L^1[0,1]^n$ (for any finite n) and those in a quotient of an algebra of analytic functions.

Let $L^1((\mathfrak{R}^+)^n)$ be the Banach algebra of integrable functions on $(\mathfrak{R}^+)^n$ with the usual norm and convolution as multiplication. Notice that $L^1[0,1]^n \cong L^1((\mathfrak{R}^+)^n)/I$, where I is the closed ideal in $L^1((\mathfrak{R}^+)^n)$ of functions whose support is contained in the complement of $[0,1]^n$. Define $A_0^{(n)}$ as the Banach algebra of functions of n complex variables which are continuous on the n-fold Cartesian product of the closed right half-plane, analytic on the interior of this set, and which vanish at infinity. The Laplace transform is a continuous monomorphism of $L^1((\mathfrak{R}^+)^n)$ into (but not onto) $A_0^{(n)}$. Let $\mathbf{K}$ be the ideal $e^{-z_1}A_0^{(n)} + \cdots + e^{-z_n}A_0^{(n)}$. A function f in $L^1((\mathfrak{R}^+)^n)$ is in I if and only if $\mathfrak{L}(f)$ is in the closure of $\mathbf{K}$ (see Section 4). Thus, $\mathfrak{L}$ induces a continuous monomorphism $\tilde{\mathfrak{L}}$ from $L^1[0,1]^n$ into $A_0^{(n)}/\bar{\mathbf{K}}$. If M is any closed ideal in $A_0^{(n)}/\bar{\mathbf{K}}$ then $\tilde{\mathfrak{L}}^{-1}(M)$ is a closed ideal in $L^1[0,1]^n$. We prove (Theorem 4.6) that $\tilde{\mathfrak{L}}^{-1}$ actually implements a bijection between closed ideals in $L^1[0,1]^n$ and $A_0^{(n)}/\bar{\mathbf{K}}$, so that an ideal J in $L^1[0,1]^n$ is dense if and only if $\tilde{\mathfrak{L}}(J)$ is dense in $A_0^{(n)}/\bar{\mathbf{K}}$.

In 1950 Nyman proved that ideals in $L^1(\mathfrak{R}^+)$ are dense if and only if their image under the Laplace transform is dense in $A_0^{(1)}$ ([8], [3]). In 1981 Domar showed that under suitable conditions on a weight w, all closed ideals in $L^1(\mathfrak{R}^+, w)$ are of the form M_β (defined above) [4]. (It can be shown, using recent results of Thomas [10], that there are weights w on $\mathfrak{R}^+$ for which this is not true.) The key idea in both of these results is to show that functions belonging to the annihilator of certain ideals necessarily have compact support. It is reasonable to expect that some of our methods should be useful in extending Domar's and Nyman's results to higher dimensions.

There are two problems in function theory which are relevant here. It is not difficult to see that the ideal $\mathbf{K} = e^{-z_1}A_0^{(1)}$ is closed in $A_0^{(1)}$. Is the ideal $\mathbf{K}$ closed in $A_0^{(n)}$ for $n>1$?

The second problem is that of finding a characterization of dense ideals in $A_0^{(n)}$ or $A_0^{(n)}/\bar{\mathbf{K}}$ for $n>1$. In Section 3 we discuss certain examples of nondense

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