SOME REMARKS ON POSITIVELY CURVED 4-MANIFOLDS

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1. The aim of this note is to prove the following.

THEOREM. *Let M^4 be a compact, connected, oriented, positively curved Riemannian 4-manifold without boundary. Then M admits at most one harmonic 2-form of constant length* (*up to constant multiples*). *If M admits such a 2-form, then M is definite.*

Background and motivation for the above theorem can be summarized as follows: the Hopf conjecture asserts that $\mathbf{S}^2 \times \mathbf{S}^2$ admits no metric of strictly positive sectional curvature. We have attempted to gain insight into this conjecture by starting with a positively curved compact, oriented 4-manifold M (with the metric normalized so that the sectional curvature K satisfies $1 \geq K \geq \delta > 0$). From Synge's theorem, M is simply connected. Since M is 4-dimensional, it follows that $H_1(M, \mathbf{Z}) = H_3(M; \mathbf{Z}) = 0$ and $H_2(M; \mathbf{Z}) \cong H^2(M; \mathbf{Z})$ is torsion-free. It follows from [2] that we know M topologically once we know its intersection form.

From a (Riemannian) geometric point of view we know

$$H^2(M; \mathbf{R}) = H^2(M; \mathbf{Z}) \otimes \mathbf{R}$$

as the DeRham cohomology and as the space of harmonic 2-forms (relative to the subsumed metric). It seems natural to ask, then, if there are topological restrictions to the "types" of harmonic 2-forms such a manifold can admit? In [3] we showed that if our M admits a *parallel* 2-form, then $\dim H^2(M; \mathbf{R}) = 1$ and it follows that M is $\mathbf{CP}^2$ (topologically and even biholomorphically).

In the theorem above we relax the assumption of the existence of a parallel 2-form to the existence of a harmonic 2-form of constant length.

While this assumption is clearly quite strong, it is strictly weaker than parallel, and we can at least still conclude that M is definite, so that we conclude that a smooth topological indefinite four manifold can never support such a metric. As $\mathbf{S}^2 \times \mathbf{S}^2$, and in fact

$$\underbrace{\mathbf{S}^2 \times \mathbf{S}^2 \# \cdots \# \mathbf{S}^2 \times \mathbf{S}^2}_{n} \# \underbrace{\mathbf{CP}^2 \# \cdots \# \mathbf{CP}^2}_{m} \# \underbrace{\overline{\mathbf{CP}^2} \# \cdots \# \overline{\mathbf{CP}^2}}_{k}$$

where either $n > 0$, or $n = 0$ and $m \cdot k \neq 0$, are all indefinite, our theorem rules out all such manifolds. In fact, it follows from [1] and [2] that a definite, smooth simply connected compact 4-manifold must be topologically

$$\underbrace{\mathbf{CP}^2 \# \cdots \# \mathbf{CP}^2}_{b_2}.$$

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