HOMOMORPHISMS BETWEEN ALGEBRAS OF DIFFERENTIABLE FUNCTIONS IN INFINITE DIMENSIONS

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Introduction. Let E and F be real Banach spaces. For $n = 0, 1, \ldots, \infty$, let $C^n_{wub}(E;F)$ be the space of n-times continuously differentiable functions f: $E \to F$ such that, for each integer $j \leq n$ and each $x \in E$, both the jth derivative mapping $f^j : E \to P(^jE;F)$ and the polynomial $f^j(x) \in P(^jE;F)$ are weakly uniformly continuous on bounded subsets of E. (This space and related notions are reviewed below.) Our primary interest here is in the study of homomorphisms $A: C^n_{wub}(E;R) \to C^m_{wub}(F;R)$. We will show that these homomorphisms are induced by functions $g: F'' \to E''$, in a way to be made more precise later. One of the principal purposes of this note is to characterize these functions g in terms of a differentiability property, thereby characterizing the homomorphisms A. An easy consequence will be that every such homomorphism is automatically continuous when the spaces C^n_{wub} are given their natural topology.

By way of defending our interest in $C^n_{wub}(E;F)$, we mention that several quite natural characterizations of this space exist, and are recalled below. In particular, if $E = R^n$ and $F = R^k$ then weak uniform continuity on bounded sets is automatic. In other words, in the case of finite-dimensional spaces E and F, our results reduce to the classical case of homomorphisms $A: C^n(R^q) \to C^m(R^k)$; this is stated as Corollary 3.5 below. (See Glaeser [8] and Bers [4] for discussions of related problems in finite-dimensional real and complex normed spaces, respectively.) Moreover, complex analogs of this space are of independent interest and of some relevance to the Michael problem on automatic continuity of complex-valued homomorphisms on a complex Fréchet algebra. For example, let $E = c_0$ be considered as a complex Banach space, let $F = C$ and $n = \infty$, and call the corresponding space $H_{wub}(c_0)$. Then it is known [5] that if every scalar-valued homomorphism on $H_{wub}(c_0)$ is continuous then every scalar-valued homomorphism on every Fréchet algebra is continuous.

The basic ingredients we will need are few and are all relatively simple. First, under reasonable hypotheses (such as E' having the bounded approximation property), $C^n_{wub}(E;R)$ can be characterized as the completion of the unital algebra generated by E' under the topology of uniform convergence of a function and its first k derivatives on bounded sets, where $k \in N$ and $k \leq n$. Therefore, a continuous homomorphism $\Phi: C^n_{wub}(E; R) \to R$ is determined by its action on E'.

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