SOLVABILITY OF INVARIANT SECOND-ORDER DIFFERENTIAL OPERATORS ON METABELIAN GROUPS

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Introduction. This paper is the third in a series (see [3], [4]) devoted to the idea of applying group representations in order to obtain solvability properties of differential operators on groups and homogeneous spaces. A typical theorem in the subject asserts that if L is a left-invariant differential operator on a connected Lie group G and if its infinitesimal components $\pi(L)$, $\pi \in \hat{G}$, have bounded inverses with suitable norm-growth properties, then L is semiglobally solvable. Theorems 2.1 and 2.4 of [3] are of this variety. In [4] these theorems are improved by allowing for Sobolev-type norms on the representation spaces. The main theorem of [4] is then applied to prove new solvability results for differential operators on the affine motion group of the line (i.e., the $ax+b$ group). In this paper we broaden the applications of [4] considerably.

Until [3], virtually all group representations-related work on solvability concentrated on *nilpotent* groups and transversally elliptic operators. The framework for much of my work in solvability has been *solvable* groups $G=SN$ which are semidirect products of a normal simply connected nilpotent subgroup N and a vector group S. Within this framework I have considered parabolic and hyperbolic operators in addition to transversally elliptic operators. For example, a typical operator of interest is the variable-coefficient heat operator $\mathcal{H}=\sum_i A_i-\sum_j X_j^2$, where $\{A_i\}$ is a basis of the Lie algebra $\mathfrak{s}$ of S, and $\{X_j\}$ is a basis of $\mathfrak{n}$. If we set $A=\sum_i A_i$ and consider its span, we see it is really no loss of generality to assume that $\dim S=1$. In this paper we study parabolic and hyperbolic second-order operators on $G=SN$ where N is *abelian* and $\dim S=1$. (The $ax+b$ group is the simplest noncommutative example of such a group.) So let $A\in\mathfrak{s}$, $A\neq 0$, and let $\{X_j\}_{j=1}^r$ be a basis of $\mathfrak{n}$. Denote the Laplacian on $\mathfrak{n}$ (with respect to this basis) by $\Delta=\sum_{j=1}^r X_j^2$. We shall consider the following three "classical" second-order operators:

(i) Heat Operator $\mathcal{H}=A-\Delta$,

(ii) Schroedinger Operator $\mathcal{S}=iA-\Delta$,

(iii) Wave Operator $\mathcal{W}=A^2-\Delta$.

We saw in [3] and [4] that the eigenvalues of the matrix $\Lambda=\mathrm{Ad}_{\mathfrak{n}} A$ play a critical role in solvability results on metabelian solvable groups. They continue to play a feature role here; in fact, our main result is the following.

THEOREM. *Suppose that all the eigenvalues of Λ have positive real part. Then each of the operators $\mathcal{H}$, $\mathcal{S}$, $\mathcal{W}$ is globally solvable.*

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