NOTE ON A PAPER OF P. PHILIPPON

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In Proposition 3.3 of [6], Philippon has sharpened previous results of Masser and Wüstholz on the degrees of the isolated components of ideals generated by polynomials of known degree over a field of characteristic 0. Here we propose to shorten and, we hope, render the scheme of that proof more transparent by (a) returning to the systematic use of localization (as in [2], [3], [4]) and (b) by defining a convolution to state the multihomogeneous Bezout theorem and to evaluate the highest homogeneous term of the Hilbert polynomial. The first half of Philippon's proposition would also fit neatly into our framework, but a concise version was given in [1] already.

We need some notation. A multihomogeneous ideal in $R=k[\mathbf{x}_1,\dots,\mathbf{x}_p]$ is an ideal I generated by polynomials which are simultaneously homogeneous in each of the p sets of variables $\mathbf{x}_i=(x_{i0},\dots,x_{iN_i})$ separately. If I is also prime and no $\mathbf{x}_i\subset I$, then I is called *relevant*. For $d\geq 0$, let

$$\mathfrak{N}(d)=\{\mathbf{j}=(j_1,\dots,j_p)\in\mathbf{Z}^p_{\geq 0}: j_1+\cdots+j_p=d\}.$$

For $\mathbf{D}=(D_1,\dots,D_p)$ and $\boldsymbol{\delta}=(\delta_{\mathbf{j}})_{\mathbf{j}\in\mathfrak{N}(d)}$, define the convolution $\boldsymbol{\pi}=\boldsymbol{\delta}*\mathbf{D}$ by the formula $\pi_{\mathbf{j}}=\sum_{i=1}^{p}D_i\delta_{\mathbf{j}+\mathbf{e}_i}$ for each $\mathbf{j}\in\mathfrak{N}(d-1)$, where $\mathbf{e}_1,\dots,\mathbf{e}_p$ are the standard basis of $\mathbf{Z}^p$. We note that if $\dim I=d\geq 0$, then I has a degree $\boldsymbol{\delta}(I)$ with components $\delta_{\mathbf{j}}(I)\in\mathbf{Z}_{\geq 0}$ for every $\mathbf{j}\in\mathfrak{N}(d)$ and some $\delta_{\mathbf{j}}(I)$ positive. Moreover when $d\geq 1$, the multihomogeneous Bezout theorem (see, e.g., Lemma A5 of [4]) states that if P is multihomogeneous of degree $\mathbf{D}$ and P lies in no associated prime of I, then (I,P) is multihomogeneous of dimension $d-1$ and its degree can be expressed in our notation as $\boldsymbol{\delta}(I,P)=\boldsymbol{\delta}(I)*\mathbf{D}$.

For a fixed multihomogeneous ideal $\mathfrak{A}$, Philippon works in the open set U of the maximal spectrum M consisting of those maximal (relevant) ideals, $\mathfrak{M}\in M$, such that $\mathfrak{M}\not\supset\mathfrak{A}$. We say that a multihomogeneous ideal J is *U-perfect* if for every $\mathfrak{M}\in U$, the ring $R_{\mathfrak{M}}/JR_{\mathfrak{M}}$ is Cohen–Macaulay. Then, if $\mathfrak{P}$ is a multihomogeneous prime ideal in such an $\mathfrak{M}$, $R_{\mathfrak{P}}/JR_{\mathfrak{P}}$ is also Cohen–Macaulay. For any multihomogeneous ideal I, a primary component contained in an $\mathfrak{M}\in M$ will be called a primary *U-component* of I, and similarly for associated prime components. Philippon introduced a function $S_UH(I;\mathbf{D})$, which can be written in our notation as

$$S_UH(I;\mathbf{D})=\sum_{k=0}^{d}\delta_k(I_{(k)})*\underbrace{\mathbf{D}*\cdots*\mathbf{D}}_{k\text{ times}}=\sum_{k=0}^{d}\boldsymbol{\delta}(I_{(k)})*\mathbf{D}^k,$$

where $I_{(k)}$ is the intersection of all isolated U-components of I of dimension k. The *U-degree* (*U-dimension*) of an ideal is the degree (dimension) of the inter-

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