HILBERT TRANSFORM IN THE COMPLEX PLANE AND AREA INEQUALITIES FOR CERTAIN QUADRATIC DIFFERENTIALS

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Introduction. In this paper we are concerned with a singular integral of Calderon–Zygmund type defined for functions of one complex variable as

$$(Tf)(z) = -\frac{1}{\pi}\iint \frac{f(\zeta)\,d\mu(\zeta)}{(z-\zeta)^2},$$

where $d\mu(\zeta)$ denotes the Lebesgue measure in $\mathbf{C}$. This integral is known as the Hilbert transform in the complex plane or the Ahlfors–Beurling transform. Its Fourier multiplier is

$$m(\xi) = \frac{(Tf)^{\wedge}(\xi)}{f^{\wedge}(\xi)} = \frac{\xi}{\bar{\xi}}, \quad \xi \in \mathbf{C}-\{0\}.$$

In particular, T is a unitary operator in $L^2(\mathbf{C})$ and it changes the complex derivatives $\partial/\partial\bar{z}$ and $\partial/\partial z$; in symbols,

$$T\circ\frac{\partial}{\partial\bar{z}} = \frac{\partial}{\partial z}. \tag{1}$$

This fundamental property of the Hilbert transform has led to various applications to the plane quasiconformal mappings and the theory of systems of partial differential equations in complex variables.

We are specifically concerned with the Hilbert transform of the characteristic function χ_E of a measurable subset E in the unit disk $\mathbf{B} = \{z : |z| < 1\}$. A weak (1.1)-type inequality shows that

$$\iint_{\mathbf{B}} |T\chi_E(z)|\,d\mu(z) \le A|E|\log\frac{\pi}{|E|} + C|E|, \tag{2}$$

where $|E|$ stands for the Lebesgue measure of E. The constants A and C are independent of the set E.

In 1966, Gehring and Reich [2] recognized that the best possible constant A in (2) is strictly related to the degree of regularity of a quasiconformal mapping. This constant is expected to be equal to one. Reich [12; 13] succeeded in proving that $A \le 17$.

In this paper we wish to treat various cases where we have reached $A = 1$ in (2). The main results, however, concern certain sharp estimates for hyperelliptic differentials which are of independent interest. Other papers concerned with sharp inequalities for the Hilbert transform are [11], [3], and [4].

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