ON FRACTIONAL DERIVATIVES AND STAR INVARIANT SUBSPACES

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1. Introduction and statement of main results. Let $\phi(z)$ be an inner function defined on the unit disk $D=\{|z|<1\}$. Factor ϕ canonically as

$$\phi(z)=\lambda B(z)s_\sigma(z),$$

where $|\lambda|=1$,

$$B(z)=\prod_{k=1}^{\infty}\frac{\bar{a}_k}{|a_k|}\frac{a_k-z}{1-\bar{a}_k z}$$

is a Blaschke product and

$$s_\sigma(z)=\exp\left(-\int_T\frac{\zeta+z}{\zeta-z}\,d\sigma(\zeta)\right)$$

where σ is a positive singular measure on the unit circle T.

In [4] we proved the following result, extending earlier work of Frostman, Riesz, and Ahern and Clark (see [6] and [1]).

THEOREM A. *Let* $\zeta_0\in T$, $\phi=Bs_\sigma$, *and* $1<p<\infty$.

(1) *Necessary and sufficient that* $\lim_{r\to1}f(r\zeta_0)$ *exist for all* $f\in K_*(\phi)$ *is that*

$$\sum_k\frac{1-|a_k|}{|\zeta_0-a_k|}+\int_T\frac{d\sigma(\zeta)}{|\zeta_0-\zeta|}<\infty.$$

(2) *Necessary and sufficient that* $\lim_{r\to1}f(r\zeta_0)$ *exist for all* $f\in K_p(\phi)$ *is that*

$$\sum_k\frac{1-|a_k|}{|\zeta_0-a_k|^q}+\int_T\frac{d\sigma(\zeta)}{|\zeta_0-\zeta|^q}<\infty.$$

(Here and in the sequel, by $\lim_{r\to1}f(r\zeta_0)$ we mean $\lim_{r\to1-}f(r\zeta_0)$.)

The spaces $K_p=K_p(\phi)$ and $K_*=K_*(\phi)$ are the "star-invariant" subspaces of H^p and BMOA determined by

$$K_p(\phi)=\phi\bar{H}_0^p\cap H^p\quad\text{and}\quad K_*(\phi)=K_2(\phi)\cap\text{BMO},$$

where $\bar{H}_0^p=\{\bar{z}\bar{f}(z):f\in H^p\}$.

Although derivatives are not mentioned in [4] it is not difficult to conjecture (and prove) the correct results for the radial behavior of $f^{(1)},f^{(2)},\dots$ if $f\in K_p$ or K_*, and arrive at the following result.

THEOREM A′. *Let* $\zeta_0\in T$, $\phi=Bs_\sigma$, $1<p<\infty$ *and* $n=0,1,2,\dots$.

(1) *Necessary and sufficient that* $\lim_{f\to1}f^{(n)}(r\zeta_0)$ *exist for all* $f\in K_*$ *is that*

$$\sum\frac{1-|a_k|}{|\zeta_0-a_k|^{n+1}}+\int_T\frac{d\sigma(\zeta)}{|\zeta_0-\zeta|^{n+1}}<\infty.$$

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