SOME FUNCTION THEORETIC PROPERTIES OF THE GAUSS MAP FOR HYPERBOLIC COMPLETE MINIMAL SURFACES

Allen Weitsman and Frederico Xavier

Introduction. Let M be a simply connected complete minimal surface (immersed) in $\mathbf{R}^3$. It is classical that M can be parameterized by pairs (f, g) where f is analytic, g is meromorphic, and the zeros of f occur precisely at the poles of g, the order of the zero being twice that of the pole. The Weierstrass representation (cf. [9, p. 63]) of M given in terms of f and g is the parameterization

$$
\begin{aligned}
x_1(z) &= \frac{1}{2}\operatorname{Re}\int^z f(1-g^2)\,dz, \\
x_2(z) &= \frac{1}{2}\operatorname{Re} i\int^z f(1+g^2)\,dz, \qquad (1)\\
x_3(z) &= \operatorname{Re}\int^z fg\,dz.
\end{aligned}
$$

The metric and curvature are given by

$$
\begin{aligned}
\lambda(z)|dz| &= \frac{1}{2}|f|(1+|g|^2)|dz|, \qquad (2)\\
K &= -\left(\frac{4|g'|}{|f|(1+|g|^2)^2}\right)^2.
\end{aligned}
$$

An important feature of g is that, after composition with stereographic projection, it represents the Gauss map of the surface. The universal covering surface of a hyperbolic minimal surface is a simply connected surface conformally equivalent to the unit disk $\mathbf{D}$, and can therefore be given as above, where the parameter space is the unit disk. In particular, if the surface itself is simply connected we can and do take f and g as defined in $\mathbf{D}$. The completeness condition then means that $\int_\alpha \lambda|dz| = \infty$ for every path α tending to $\partial\mathbf{D}$.

A fundamental problem in the theory of complete minimal surfaces is the determination of which meromorphic functions g arise as Gauss maps of these surfaces. It is known that if g is holomorphic it cannot be in the Nevanlinna class [6, pp. 394–5] and that g cannot omit seven points [12]. In the present note we shall give some further restrictions.

It is perhaps important to point out that in [12], as well as in Theorems 1 and 2 below, an essential ingredient in the proofs is a general result of Yau [13, p. 661] for complete Riemannian manifolds. It is interesting that, although in the present

---

Received May 13, 1986. Revision received September 8, 1986.
Research of the authors was supported by the National Science Foundation.
Michigan Math. J. 34 (1987).