TRANSVERSAL JACOBI FIELDS FOR HARMONIC FOLIATIONS

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1. Introduction. A foliation $\mathfrak{F}$ on a manifold M is given by the exact sequence of vectorbundles

$$0 \to L \to TM \xrightarrow{\pi} Q \to 0,$$

where L is the tangent bundle and Q the normal bundle of $\mathfrak{F}$. If $V(\mathfrak{F})$ denotes the Lie algebra of infinitesimal automorphisms of $\mathfrak{F}$, we have an exact sequence of Lie algebras

$$0 \to \Gamma L \to V(\mathfrak{F}) \xrightarrow{\pi} \Gamma Q^L \to 0,$$

where ΓQ^L denotes the invariant sections of Q under the action of ΓL by Lie derivatives [4; 9]. We assume throughout that $\mathfrak{F}$ is Riemannian, with a bundle-like metric g_M on M inducing the holonomy invariant metric g_Q on $Q \cong L^\perp$ [10]. ∇ denotes the unique metric and torsion-free connection in Q (see, e.g., [3; 9; 10]). Associated to ∇ are transversal curvature data, in particular the (transversal) Ricci operator $\rho_\nabla : Q \to Q$ and the Jacobi operator $J_\nabla = \Delta - \rho_\nabla : \Gamma Q \to \Gamma Q$ [4]. In this paper we study geometric properties of infinitesimal automorphisms $Y \in V(\mathfrak{F})$ such that $\bar{Y} = \pi(Y) \in \Gamma Q^L$ satisfies the Jacobi condition $J_\nabla \bar{Y} = 0$. In view of the variational meaning of J_∇ [4], it is then natural to assume $\mathfrak{F}$ to be harmonic; that is, all leaves of $\mathfrak{F}$ are minimal submanifolds [3].

THEOREM A. *Let $\mathfrak{F}$ be a transversally orientable harmonic Riemannian foliation on a compact orientable Riemannian manifold (M, g_M), and Y an infinitesimal automorphism of $\mathfrak{F}$. Then the following properties are equivalent:*

(i) *$\bar{Y}$ is a transversal Killing field, that is, $\theta(Y)g_Q = 0$;*
(ii) *$\bar{Y}$ is a transversally divergence-free Jacobi field;*
(iii) *$\bar{Y}$ is transversally affine, that is, $\theta(Y)\nabla = 0$.*

REMARKS. (1) If $\mathfrak{F}$ is given by the fibers of a harmonic submersion $f: M \to N$, the equivalence of (i) and (ii) specializes to the statement that a projectable vector field $v = V \circ f$ ($V \in \Gamma TN$) along f is a divergence-free Jacobi field along f if and only if V is a Killing vector field on N (see [12] for the particular case $N = S^n$). Note, however, that the Jacobi condition for v in the harmonic map theory uses the pull-back of the Riemannian connection of N which, in general, differs from the canonical connection ∇ in Q [3].

(2) For the foliation of M by points, the equivalence of (i) and (ii) is the classical characterization of Killing vector fields given by Lichnerowicz [7] and Yano [14]; the implication (iii)$\Rightarrow$(i) is due to Yano [14].

Received April 4, 1986.

The work of the first two authors was supported in part by a grant from the National Science Foundation.

Michigan Math. J. 34 (1987).