THE REFLEXIVITY OF CONTRACTIONS WITH NONREDUCTIVE *-RESIDUAL PARTS

Katsutoshi Takahashi

Let T be a contraction on a separable Hilbert space, and suppose that there exist an isometry V $(\neq 0)$ and a (bounded linear) operator Y with dense range such that $YT = VY$, which means that the contraction T is not of class $C_{0\cdot}$, that is, $\lim_{n\to\infty}\|T^n x\| \neq 0$ for some x (cf. [6, Proposition II.3.5]). If V is non-unitary, then it is easily seen that every point in the open unit disc $D = \{\lambda : |\lambda| < 1\}$ is an eigenvalue of T^*, and so T has many invariant subspaces. It was proved in [2] that T is even reflexive in this case. But, in the case in which V is unitary, it is not yet known whether such a T always has a nontrivial invariant subspace (cf. [8]). In a recent paper [5], Kérchy has proved that if V is a bilateral shift then T has a nontrivial invariant subspace, and under the additional assumption that T is of class C_{11} (i.e., $\lim_{n\to\infty}\|T^n x\| \neq 0$ and $\lim_{n\to\infty}\|T^{*n}x\| \neq 0$ for every nonzero x), T is reflexive. The purpose of the present note is to prove a reflexivity theorem which extends these results.

For an operator T, let $\operatorname{Alg} T$ denote the weakly closed algebra generated by T and the identity I. Let $\operatorname{Lat} T$ and $\operatorname{Alg}\operatorname{Lat} T$ denote the lattice of all invariant subspaces for T and the algebra of all operators A such that $\operatorname{Lat} T \subseteq \operatorname{Lat} A$, respectively. Recall that T is reflexive if $\operatorname{Alg} T = \operatorname{Alg}\operatorname{Lat} T$.

THEOREM. *If T is a contraction on a separable Hilbert space and there exists an operator Y with dense range such that $YT = WY$ for some bilateral shift W $(\neq 0)$, then T is reflexive.*

The proof of [1, Theorem 5] shows that in the proof of our Theorem it suffices to consider the case where T is completely non-unitary, that is, where T has no nonzero invariant subspace on which it acts as a unitary operator.

Let T be a completely non-unitary contraction. We use the functional model of Sz.-Nagy and Foiaş [6] for T. Let Θ be the characteristic function of T; thus Θ is an operator-valued H^∞-function on the unit circle ∂D whose values are contractions from $\mathfrak{D}$ to $\mathfrak{D}_*$, where $\mathfrak{D} = (\operatorname{ran}(I - T^*T))^-$ and $\mathfrak{D}_* = (\operatorname{ran}(I - TT^*))^-$. We set

$$\Delta(\zeta) = (I - \Theta(\zeta)^*\Theta(\zeta))^{1/2} \quad \text{and} \quad \Delta_*(\zeta) = (I - \Theta(\zeta)\Theta(\zeta)^*)^{1/2}$$

for $\zeta \in \partial D$ and consider T being defined on the space

$$H(\Theta) = [H^2(\mathfrak{D}_*) \oplus (\Delta L^2(\mathfrak{D}))^-] \ominus \{\Theta h \oplus \Delta h : h \in H^2(\mathfrak{D})\}$$

by

$$T(f \oplus g) = P(\chi f \oplus \chi g), \tag{1}$$

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