A COVERING LEMMA FOR MAXIMAL OPERATORS WITH UNBOUNDED KERNELS

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I. Introduction. Calderon and Zygmund [1] proved that certain maximal operators are bounded on $L^p(\mathbf{R}^n)$ for $p>1$, using the rotation method. It is unknown whether they take $L^1(\mathbf{R}^n)$ into Weak $L^1(\mathbf{R}^n)$. We prove a positive result for a certain subclass of these operators. The method is to prove an analog of the usual covering lemma [4], even though the kernels are unbounded.

More specifically, let $g(\theta)$ be a positive, integrable, decreasing function on the interval $(0,1)$ such that $\theta g(\theta)$ is increasing. For $(x_1,x_2)=x\in\mathbf{R}^2$, set

$$\Omega(x)=\begin{cases} g(x_2/x_1) & \text{if } 0<x_2<x_1 \text{ and } |x|\le 1,\\ 0 & \text{otherwise.}\end{cases}$$

For $r>0$, let $\Omega_r(x)=r^{-2}\Omega(x/r)$. Define, for $f\in L^1(\mathbf{R}^2)$,

$$M_\Omega f(x)=\sup_{r>0}(\Omega_r*|f|)(x)=\sup_{r>0}\int_{\mathbf{R}^2}\Omega_r(x-y)\,|f(y)|\,dy.$$

THEOREM. *M_Ω is weak-type $(1,1)$. That is, there is a constant C such that, for every $f\in L^1(\mathbf{R}^2)$ and every $\alpha>0$,*

$$|\{x\in\mathbf{R}^2 = M_\Omega f(x)>\alpha\}|\le\frac{C}{\alpha}\|f\|_{L^1}\|g\|_{L^1}.$$

There is a similar result on $\mathbf{R}^n$, $n>2$, if $\theta g(\theta)$ is replaced by $\theta^{n-1}g(\theta)$ and $g(x_2/x_1)$ is replaced by $g(|x-(x_1,0,0,\dots,0)|/x_1)$, for $|x|\le 1$. Soria has proved such a result without restriction on $\theta g(\theta)$, but with a stronger size condition than $g\in L^1$ [3]. The idea of the proof is to use a covering lemma. However, the usual type of covering lemma does not apply because Ω may be an unbounded function. We will use the following substitute.

DEFINITION. $\Omega\in L^1(\mathbf{R}^2)$ has the selection property with constant C if, given any positive continuous function $r(x)$ defined on a measurable set $D\subseteq B_1(0)$, the unit ball of $\mathbf{R}^2$, there is a measurable subset $E\subseteq D$ such that

$$|E|\ge\frac{1}{2}|D|, \tag{1}$$

$$S(E,\Omega,r)(y)\equiv\int_E\Omega_{r(x)}(x-y)\,dx\le C \quad \text{for almost every } y\in\mathbf{R}^2. \tag{2}$$

Here, $|E|$ denotes the Lebesgue measure of E.

LEMMA. *If Ω has the selection property with constant C, then M_Ω is weak-type $(1,1)$.*

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