THE SAMELSON SPACE OF A FIBRATION

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Introduction. If G is a compact connected Lie group, then there exists a real graded vector space P_G such that $\Lambda(P_G)\cong H^*(G;\mathbf{R})$, where Λ denotes the exterior algebra. Moreover, if G acts smoothly on a connected manifold M, then there is a graded subspace $P\subset P_G$ and an algebra isomorphism $A\otimes\Lambda(P)\cong H^*(M;\mathbf{R})$ which makes the following diagram commutative:

$$\begin{array}{ccc} A\otimes\Lambda(P) & \longrightarrow & \Lambda(P_G) \\ \downarrow\cong & & \downarrow\cong \\ H^*(M;\mathbb{R}) & \xrightarrow{\omega^*} & H^*(G;\mathbb{R}) \end{array}$$

The map ω^* is induced by the orbit map $\omega: G\to M$, $\omega(g)=g\cdot x$ (for fixed $x\in M$) and $A\otimes\Lambda(P)\to\Lambda(P_G)$ denotes projection onto $\Lambda(P)$. (See [12, p. 312]).

The action of G on M gives rise to the Borel fibration $M\to MG\to BG$, and it is well known that the orbit map ω corresponds to the "transgression" $\partial:\Omega BG\to M$ via the homotopy equivalence $\Omega BG\simeq G$. The commutative diagram above then provides an isomorphism $\operatorname{Im}\partial^*\cong\Lambda(P)$.

Because these notions are extensions of the classical Lie theoretic approach of Samelson, we say that P is the Samelson subspace of the action.

It is natural to ask if analogous results hold for arbitrary fibrations $F\to E\to B$ and the associated "action" $F\times\Omega B\to F$. This question was answered in [16], where it was shown that F has a rational decomposition $\mathfrak{F}\times K$ with $K\subset\Omega B$ and $H^*(K)\cong\operatorname{Im}(\partial^*: H^*(F)\to H^*(\Omega B))$. The space K is called the *Samelson space* of the fibration because of the obvious analogy to the classical result stated earlier. In fact, the classical theorem is simply a special case of the rational decomposition described above.

The purpose of this paper is to present rational versions of various topological results within the unifying framework of the Samelson space method. In particular, we obtain an elementary proof of the Transgression Theorem [3] and a generalization of the Allday–Halperin inequality [1].

The main result of [16] forms the starting point for this paper, so we recall it in Section 1. Although minimal model theory was the fundamental tool of [16], it shall not be emphasized here. It is hoped that, by stating the results of this paper in customary topological language, a wider audience will be introduced to the efficacy of the Samelson space technique. Furthermore, with the exception of some results on rational holonomy [6] and on elliptic spaces ([13]; [4]), all the ingredients for the results of this paper were present years ago. It seems only right, then, to approach this work in the spirit of classical homotopy theory.

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