ON ANALYTIC FUNCTIONS WITH CLUSTER SETS OF FINITE LINEAR MEASURE

Ch. Pommerenke

1. Introduction. Let f be a non-constant complex-valued function defined in the unit disk $\mathbf{D}$. The total cluster set $C(f)$ consists of all limit points of $f(z)$ as $|z|\to 1$, $z\in\mathbf{D}$. The linear Hausdorff measure of $E\subset\mathbf{C}$ is defined by

$$\Lambda(E)=\lim_{\epsilon\to 0}\inf_{(D_n)}\sum_n \operatorname{diam} D_n \tag{1.1}$$

where the infimum is taken over all systems (D_n) of disks with $\operatorname{diam} D_n<\epsilon$ that cover E.

THEOREM. *If f is bounded and analytic in $\mathbf{D}$ and if*

$$\Lambda(C(f))<\infty, \tag{1.2}$$

then f has a continuous extension to $\bar{\mathbf{D}}$.

This result was proved by Globevnik and Stout [4, Theorem 2] under the additional assumption that

$$\iint_{\mathbf{D}} |f'(z)|^2\,dx\,dy<\infty, \tag{1.3}$$

and they conjectured that (1.3) is redundant. They applied their result to study proper analytic maps of $\mathbf{D}$ into the unit ball of $\mathbf{C}^N$; see [2] and [3] for related results. I want to thank Professor Globevnik for writing to me about this problem.

Note that (1.2) and (1.3) do not imply ([4], [5]) that f' belongs to the Hardy space H^1. See [5] for further results that follow from (1.2).

2. Auxiliary results. In the following lemma, it is probably possible to replace the factor π by 2.

LEMMA 1. *If B is a continuum with $\Lambda(B)<\infty$ and if V_j are the bounded components of $\mathbf{C}\setminus B$, then*

$$\sum_j \Lambda(\partial V_j)\le \pi\Lambda(B). \tag{2.1}$$

Proof. In each component V_j we fix a point w_j. By (1.1) the compact set B can be covered by finitely many disks $D_{n\mu}$ $(\mu=1,\dots,m_n)$ such that

$$\sum_{\mu=1}^{m_n} \operatorname{diam} D_{n\mu}<\Lambda(B)+\frac{1}{n}\quad\text{for } n=1,2,\dots, \tag{2.2}$$

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