# HOLOMORPHIC FUNCTIONS ON THE POLYDISC HAVING POSITIVE REAL PART

John N. McDonald

Let $D^N$ denote the open unit polydisc in $C^N$ and let

$$\mathcal{P}_N = \{f \mid f \text{ is holomorphic on } D^N,\ \text{Re} f > 0, \text{ and } f(\theta) = f(0,0,\dots,0) = 1\}.$$

Of course $\mathcal{P}_N$ is compact in the topology of uniform convergence on compacta. Thus, it follows from the Krein–Milman theorem that $\mathcal{P}_N$ is the closed convex hull of its extreme elements. In the case $N=1$ the extreme elements of $\mathcal{P}_N$ are easily found via Herglotz's theorem. For $N>1$, however, a complete description of the extreme elements of $\mathcal{P}_N$ is not known, although Forelli has found a necessary condition for a member of $\mathcal{P}_N$ to be extreme. (See [1].) Forelli [1] and McDonald [3; 4] have also constructed several examples of extreme elements of $\mathcal{P}_2$.

In this paper, we study certain faces of the convex set $\mathcal{P}_N$. We recall that a face $F$ of a convex set $S$ is a convex subset which satisfies: $(c,x,y) \in (0,1)\times S\times S$ and $cx+(1-c)y \in F$ together imply $x, y \in F$. For our purposes, it is important to note that an extreme point of the face $F$ is also an extreme point of $S$. For each $x \in S$ there is a smallest face $\mathfrak{F}(x)$ containing $x$. $\mathfrak{F}(x)$ is simply the union of all line segments from $S$ which contain $x$ as a relative interior point. If $S$ is a compact convex subset of some locally convex vector space, then the closed faces will always contain extreme elements. Faces of the form $\mathfrak{F}(x)$ are, however, not closed in general, but, if it can be shown that $\mathfrak{F}(x)$ is finite-dimensional, then $\mathfrak{F}(x)$ will necessarily be closed. Furthermore, if it is known that $\mathfrak{F}(x)$ is finite-dimensional, then it follows from a theorem of Carathéodory that $x$ can be written as a finite convex combination of extreme elements of $S$. (See, e.g., [5].)

Our main result is that $\mathfrak{F}(G)$ is a finite-dimensional face of $\mathcal{P}_N$ when $G$ is of the form $G=(1+g)/(1-g)$, where $g$ is a rational inner function satisfying $g(\theta)=0$. We also show that each member of $\mathfrak{F}(G)$ is the Cayley transform of a rational inner function and that the set of extreme elements of sets of the form $\mathfrak{F}(G)$ is dense in the set of extreme elements of $\mathcal{P}_N$. Finally, we study some particular examples of faces of the form $\mathfrak{F}(G)$.

**1. The main result.** In this section $g$ will denote a rational inner function on $D^N$ which satisfies $g(\theta)=0$. It is known that $g$ must have the form

$$g = MQ^*/Q, \tag{1}$$

where $Q$ is a polynomial having no zero on $D^N$, where

$$Q^*(z) = Q^*(z_1,\dots,z_N) = \overline{Q(1/\bar{z}_1,\dots,1/\bar{z}_N)},$$

Received January 21, 1986. Revision received June 17, 1986.
Michigan Math. J. 34 (1987).