HOLOMORPHIC VECTOR FIELDS ON COMPLEX MANIFOLDS

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1. Introduction. Let M^n be a compact complex manifold of complex dimension n, and let $\mathrm{Hol}(M)$ denote the biholomorphisms of M^n. It is known [2] that $\mathrm{Hol}(M)$ is a Lie group which, in general, is not compact. The identity component $\mathrm{Hol}(M)^0$ is generated by vector fields which, when "transferred" to M^n, are holomorphic—that is, locally of the form $Z=\sum a_k(\partial/\partial z_k)$, where the a_k are holomorphic functions. More precisely, if ψ_t is a one-parameter subgroup of $\mathrm{Hol}(M)$ then its generator X is an infinitesimal automorphism of the complex structure, and Z is holomorphic exactly when $Z=X-iJX$ for such a real vector field X. Here J denotes the complex structure. If the subgroup ψ_t has no fixed points then X (and hence Z) is nonsingular.

It was shown by Matsushima [11], using Blanchard's theorem [1] on projective embeddings, that if M^n is projective algebraic with first Betti number $b_1(M)=0$ then M^n admits no nonsingular holomorphic vector field. This result was extended to all compact Kähler manifolds by Carrell and Lieberman [4], and by Sommese [12]. In this paper we extend the theorem to more general complex manifolds, and prove an analogue for all compact complex manifolds that is similar to a result of Bott [3].

To formulate the simplest of our results we recall the definition of the Hodge numbers $h^{p,q}(M)=\dim_{\mathbf{C}} H^q(M,\Omega^p)$. It is known [5] that the Euler characteristic

$$\begin{aligned}\chi(M^n)&=\sum_{0\le p,q\le n}(-1)^{p+q}h^{p,q}(M)\\ &=2+\sum_{0<p+q=\text{even}<2n}h^{p,q}(M)-\sum_{p+q=\text{odd}}h^{p,q}(M).\end{aligned}$$

Consequently, if $\sum_{p+q=\text{odd}}h^{p,q}=0$ then, according to the well-known theorem of Poincaré–Hopf, M^n admits no nonsingular vector field. For holomorphic vector fields we have the following refinement.

THEOREM A. *Let M^n be an n-dimensional compact complex manifold. If $\sum_{0\le p\le n-1}h^{p,p+1}(M)=0$ then M^n admits no nonsingular holomorphic vector field.*

In Section 2 we prove Theorem A and in Section 3 we present two generalizations of the theorem of Carrell–Lieberman and Sommese. Section 4 treats some related results and points out some open problems.

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