AN INVARIANT FOR UNITARY REPRESENTATIONS OF NILPOTENT LIE GROUPS

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1. Introduction. In this paper, we define an invariant for the irreducible unitary representations of a simply connected nilpotent Lie group G. The invariant $i(\rho)$ for a representation ρ is an element of $H^*(\mathcal{G})$, the real cohomology of the Lie algebra $\mathcal{G}$ of G. The cohomology class $i(\rho)$ is constructed using the coadjoint orbit $\mathcal{O}$ corresponding to ρ and has degree $\dim(\mathcal{O})+1$. If we view $\dim(\mathcal{O})$ as a primary invariant, then $i(\rho)$ is a more subtle secondary invariant that can be used to distinguish between representations whose orbits have the same dimension.

The definition of $i(\rho)$ in terms of orbits is given in Section 2. The remaining sections address two central questions concerning $i(\rho)$. Firstly, is the invariant computable in examples and can it be non-zero? Secondly, what information does the invariant contain about a representation—that is, what does it measure?

Examples are presented in Section 4. These show that the invariant is relatively easy to compute and is frequently non-zero. The second question is more difficult and provides a direction for further research. Here we present three results along these lines. In Section 3, we show that if two representations differ by a multiplicative character then the invariants for these representations coincide. In Section 5, we prove that for groups with one-dimensional center, $i(\rho)$ is non-zero for representations ρ that are square integrable modulo the center. In Section 6, we show that for a class of groups (the 3-step groups with one-dimensional center), the invariant vanishes for certain representations.

In Section 7, we discuss some unsolved problems concerning $i(\rho)$.

2. Definition of the invariant. We begin with the symplectic structure for coadjoint orbits. Throughout, G will denote a Lie group with Lie algebra $\mathcal{G}$. We write $\mathcal{O}_f$ for the orbit of $f \in \mathcal{G}^*$ under the coadjoint action of G on $\mathcal{G}^*$. We have a projection map

$$\pi_f : G \to \mathcal{O}_f, \quad \pi_f(g) = \mathrm{Ad}^*(g)f. \tag{2.1}$$

If w is the 2-form on $\mathcal{O}_f$ constructed in [8], then we have

$$\pi_f^*(w) = -\,d f. \tag{2.2}$$

Here we view f as a left invariant 1-form on G, and df is the exterior derivative of f in the de Rham complex $(\Omega(G), d)$.

The left invariant forms on G are a subcomplex of $\Omega(G)$ which can be identified with the exterior algebra $\Lambda(\mathcal{G}^*)$. The cohomology of this complex is denoted by $H^*(\mathcal{G})$ and agrees with the algebraic notion of Lie algebra cohomology with trivial (real) coefficients [3].

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