ON A GEOMETRIC LOCALIZATION OF THE CAUCHY POTENTIALS

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1. Introductory remarks. Let $u \in D'(\mathbf{C})$ be a distribution in $\mathbf{C}=\mathbf{R}^2$. If h is an arbitrary C_0^∞-function in $\mathbf{C}$ (i.e., a C^∞-function with a compact support), then it is well known that the Leibniz differentiation rule still holds for the product uh (see, e.g., [12, Ch. VI]).

In particular,

$$\frac{\partial}{\partial \bar{z}}(uh)=\left(\frac{\partial}{\partial \bar{z}}u\right)\cdot h+u\left(\frac{\partial}{\partial \bar{z}}h\right),$$

where, as usual,

$$\frac{\partial}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)$$

and the equality is understood in the sense of distributions.

Let μ be a finite Borel measure in $\mathbf{C}$. The Cauchy potential (transform) $\hat{\mu}$ of μ is defined by

$$\hat{\mu}(z)=\int_{\mathbf{C}}\frac{d\mu}{\zeta-z}.$$

It is well known (see [8, Ch. II]) that $\hat{\mu}(z)$ is defined almost everywhere with respect to the area and that $\hat{\mu}(z)\in L^1_{\mathrm{loc}}(dx\,dy)$; that is, for any compact set $K\subset\mathbf{C}$,

$$\int_K|\hat{\mu}|\,dx\,dy<+\infty.$$

So $\hat{\mu}\in D'(C)$ and, as is known,

$$\frac{\partial\hat{\mu}}{\partial\bar{z}}=-\pi\mu$$

(see [7, Ch. II]; [8, Ch. II]). Thus, for all $h\in C_0^\infty$, we have

$$\frac{\partial}{\partial\bar{z}}(\hat{\mu}h)=-\pi\mu h+\hat{\mu}\frac{\partial h}{\partial\bar{z}}. \tag{1}$$

In other words,

$$\hat{\mu}h=\widehat{\mu\cdot h-\frac{1}{\pi}\hat{\mu}\frac{\partial h}{\partial\bar{z}}\,dx\,dy}.$$

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