THE SPECTRUM OF THE LAPLACIAN ON RIEMANNIAN HEISENBERG MANIFOLDS

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1. Introduction. For any compact Riemannian manifold (M, g) let $\operatorname{spec}(M, g)$ denote the collection of eigenvalues, with multiplicities, of the associated Laplace–Beltrami operator acting on $C^\infty(M)$. Two manifolds (M, g) and (M', g') are said to be isospectral if $\operatorname{spec}(M, g) = \operatorname{spec}(M', g')$. Many examples exist of pairs of isospectral, non-isometric Riemannian manifolds ([3], [6], [10], [12], [15], [17], [18]). Vigńeras gave the first examples of isospectral manifolds with non-isomorphic fundamental groups. In contrast, some manifolds such as the canonical sphere S^n and real projective space P^n, $n \leq 6$, are uniquely determined up to isometry by $\operatorname{spec}(M, g)$. (See e.g. [1], [9].)

In this paper we study the spectrum of the Laplacian of compact Riemannian Heisenberg manifolds; that is, manifolds of the form $(\Gamma \backslash H_n, g)$, where H_n is the $(2n+1)$-dimensional Heisenberg group, Γ is a uniform discrete subgroup, and g is a Riemannian metric on $\Gamma \backslash H_n$ whose lift to H_n is left-invariant. The Heisenberg manifolds are among the few manifolds for which $\operatorname{spec}(M, g)$ can be explicitly computed. By comparing the spectra of various Heisenberg manifolds, we find:

(A) If $n = 1$, $(\Gamma \backslash H_n, g)$ is uniquely determined by its spectrum.

(B) If $n > 1$, there exist many choices of pairs $(\Gamma \backslash H_n, g)$ and $(\Gamma' \backslash H_n, g')$ that are isospectral but not isometric.

More specifically, we associate with every uniform discrete subgroup Γ of H_n a positive integer denoted $|\Gamma|$. Whenever $n > 1$ and $|\Gamma| = |\Gamma'|$, there exist continuous families of metrics g_t and g_t' such that for each t, $(\Gamma \backslash H_n, g_t)$ is isospectral to $(\Gamma' \backslash H_n, g_t')$. (Note that we are *not* asserting the existence of continuous isospectral deformations of a metric.) Since $|\Gamma|$ does not always determine the isomorphism class of Γ, we thus obtain examples of isospectral manifolds with non-isomorphic fundamental groups. In some cases the manifolds are also isospectral on p-forms for all $p \geq 0$.

This paper was partly motivated by the following result of [6]. Let G be a nilpotent Lie group. In [6] we defined a group $\operatorname{AIA}(G)$ of "almost inner" automorphisms, and showed that $(\varphi(\Gamma) \backslash G, g)$ is isospectral to $(\Gamma \backslash G, g)$ for all $\varphi \in \operatorname{AIA}(G)$ whenever Γ is any uniform discrete subgroup of G and g any metric arising from a left-invariant metric on G. The manifolds are isometric if φ lies in the group $\operatorname{Inn}(G) \subset \operatorname{AIA}(G)$ of inner automorphisms but are rarely isometric otherwise. We thus obtained continuous families of non-isometric manifolds all isospectral to $(\Gamma \backslash G, g)$ under the condition $\operatorname{Inn}(G) \neq \operatorname{AIA}(G)$. We do not know whether this condition is necessary as well as sufficient for the existence of a non-

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