THE EINSTEIN-KÄHLER METRIC ON $\{|\mathbf{z}|^2+|w|^{2p}<1\}$

John S. Bland

S. Y. Cheng and S. T. Yau showed in [2] that any C^2 bounded pseudoconvex domain in $\mathbf{C}^n$ has a complete Einstein–Kähler metric with negative Ricci curvature; their solution satisfied the Monge–Ampère equation $\operatorname{Det}[\partial^2 g/\partial z_i \partial z_{\bar{j}}]=e^{(n+1)g}$, $g=\infty$ on the boundary, where the metric is given by $(\partial^2 g/\partial z_i \partial z_{\bar{j}})dz^i \otimes d\bar{z}^j$. N. Mok and S. T. Yau [4] have extended this result to arbitrary bounded pseudoconvex domains in $\mathbf{C}^n$. Explicit solutions, however, are only known in the very simplest cases. The purpose of this paper is to describe the Einstein–Kähler metric for the domain $\Omega_p=\{|\mathbf{z}|^2+|w|^{2p}<1\}$, $p>0$. These domains exhibit a wide range of boundary behavior. For $p>1$, the special boundary points $|\mathbf{z}|=1$ are C^2 weakly pseudoconvex, and the domains interpolate between B^n and $B^{n-1}\times B$. For $\frac{1}{2}<p<1$, the domains are C^1 strictly convex. For $p<\frac{1}{2}$, the boundary intersects certain real planes in cusps.

The main technique is to use the $(2n-1)$-dimensional noncompact automorphism group of Ω and the biholomorphic invariance of the Einstein–Kähler metric to reduce the Monge–Ampère equation for the metric to an ordinary differential equation in the auxiliary function $X=|w|^2/(1-|\mathbf{z}|^2)^{1/p}$. This differential equation can be easily solved to give an implicit function in X; however, all information of interest is obtained by indirect methods.

The function X contains geometric information about the domain. The leaves $X=$ constant define a real foliation of the domain, the leaves of which converge at the special boundary points $|\mathbf{z}|=1$, $w=0$. The automorphism group of the domain preserves this foliation, and acts transitively within each leaf. Thus, any biholomorphically invariant quantity can be reduced to a function of X, and it assumes its full range of values arbitrarily near the special boundary points $|\mathbf{z}|=1$; in particular, any nonconstant biholomorphically invariant quantity exhibits no limiting behavior near these boundary points.

The results of these calculations have some interesting consequences. When $p>1$, the special boundary points are C^2 weakly pseudoconvex, and the Riemannian sectional curvature for the domain is bounded between negative constants. In particular, a local Schwarz lemma can be used to obtain bounds on the metric for any domain locally approximating Ω on the inside (see Theorems 4 & 5). On the other hand, there are C^1 strictly convex domains for which the Einstein–Kähler metric has strictly positive holomorphic sectional curvature in certain directions at some points (see Theorem 4). In all cases where $p>0$, volume estimates on the Einstein–Kähler metric for locally approximating domains can be obtained (Theorem 5).

Received September 28, 1984. Revision received April 2, 1985.

Research was partially supported by the Natural Sciences and Engineering Research Council of Canada and the Institute for Advanced Study.

Michigan Math. J. 33 (1986).