ON THE ACCESSIBILITY OF THE BOUNDARY OF A SIMPLY CONNECTED DOMAIN

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Let D be a simply connected plane domain, not the whole plane, and let $w=f(z)$ map $|z|<1$ one-to-one and conformally onto D. As is well known, for almost every θ $(0\le\theta\le 2\pi)$, $f(z)$ has a finite radial limit $f(e^{i\theta})$ at $e^{i\theta}$. Consequently, the image under f of the radius at such an $e^{i\theta}$ determines an (ideal) accessible boundary point of D whose complex coordinate is $f(e^{i\theta})$ [2, pp. 357–363]. We will denote both the (ideal) accessible boundary point and its complex coordinate by $f(e^{i\theta})$; no confusion will arise provided that we treat $f(e^{i\theta_1})$ and $f(e^{i\theta_2})$ to be distinct whenever $\theta_1\neq\theta_2$ (even though the complex coordinates may be equal).

We introduce the following metric on D: the *arc-length distance* $l_D(w_1,w_2)$ between two points of D is defined to be the infimum of the Euclidean lengths of the rectifiable arcs lying in D and joining w_1 to w_2. This arc-length metric is seen to agree locally with the Euclidean metric. Let R be the set of rectifiably accessible points of ∂D. For $w\in D$ and $w_0\in R$ we let $l_D(w,w_0)$ be the infimum of the Euclidean lengths of rectifiable curves lying in D and joining w to w_0. The arc-length distance between two points of R is defined similarly. It is easily shown that l_D is a metric for $D\cup R$. The distance between two subsets S_1 and S_2 of $D\cup R$ will be denoted by $l_D(S_1,S_2)$ and is defined in the usual manner. Any limits involving elements of R will be taken using the arc-length metric.

We will let $\Delta(w,r)$ denote the open disc which is centered at w of radius r. Let $w_0\in R$. Corresponding to each positive number r small enough so that the domain D contains a disc of radius r, let

$$\delta(r,w_0)=\inf\{l_D(w,w_0):\Delta(w,r)\subseteq D\}.$$

We say that w_0 is *broadly accessible* if $\liminf_{r\to 0}\delta(r,w_0)/r=1$. In more picturesque language, $w_0\in R$ is broadly accessible if we can find discs in D close to w_0 such that the center of each disc can be joined to w_0 by an arc whose length is only slightly larger than the radius of the disc. We will use $\delta(r,\theta)$ to abbreviate $\delta(r,f(e^{i\theta}))$. Concerning the broad accessibility criterion, we will prove the following theorem.

THEOREM. *Let D be a simply connected plane domain, not the whole plane. Let f map $|z|<1$ one-to-one and conformally onto D. Then for almost every θ,*

$$\lim_{r\to 1}f(re^{i\theta})=f(e^{i\theta})$$

is a broadly accessible point of ∂D.

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