SPECTRAL THEORY OF SELF-ADJOINT HANKEL MATRICES

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The purpose of this paper is to determine, up to unitary equivalence, the absolutely continuous part of a bounded symmetric infinite Hankel matrix, in terms of the symbol of the operator. Since, according to Hartman's theorem, continuous symbols correspond to compact operators, the continuous spectrum must be connected somehow with the discontinuities of the symbol. For jump discontinuities, this is born out by the theory of the essential spectrum [6, Chapter 6] in which each discontinuity contributes a segment to the essential spectrum of length proportional to the jump.

To the author's knowledge, the only known result on multiplicity theory for the continuous spectrum is Rosenblum's work [8] on the Hilbert matrix, which he shows by explicit diagonalization to have uniform Lebesgue spectrum of multiplicity one on the interval $[0, \pi]$. Our main result gives a complete description of the absolutely continuous part in the case of symbols with a finite number of smooth jumps. We show that each discontinuity contributes a direct summand to the absolutely continuous part having uniform Lebesgue spectrum of multiplicity one, on a certain interval—the same interval that it contributes to the essential spectrum.

We shall obtain this result from a theorem first stated by Ismagilov in 1963 [3], and later proved in [2].

THEOREM (Ismagilov). *Let A and B be bounded self-adjoint operators and set $H = A + B$. If the product AB is of trace class, then the absolutely continuous part of H is unitarily equivalent to the direct sum of the absolutely continuous parts of A and B.*

This theorem is, in fact, a theorem of trace class scattering theory, generalizing the classic Kato–Rosenblum theorem on stability of absolutely continuous parts under trace class perturbations [7, p. 16]. It may be proved as a consequence of another generalization of the Kato–Rosenblum theorem due to Pearson [5, §7, p. 24], which we shall also use.

THEOREM (Pearson). *Let A be self-adjoint on $\mathcal{H}$, B self-adjoint on $\mathcal{H}'$, and J bounded from $\mathcal{H}$ to $\mathcal{H}'$. If $BJ - JA$ is trace class then the wave operator*

$$\Omega_+ = \operatorname*{s-lim}_{t \to \infty} e^{iBt} J e^{-iAt} P_a(A)$$

exists, where $P_a(A)$ is the projection onto the absolutely continuous subspace of A.

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