ON THE REFLEXIVITY OF ALGEBRAS AND LINEAR SPACES OF OPERATORS

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This paper is dedicated to our good friend George Piranian on the occasion of his retirement

Let $\mathcal{H}$ be a complex Hilbert space (of arbitrary dimension), and let $\mathcal{L}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. Among the useful topologies on $\mathcal{L}(\mathcal{H})$ are the weak* topology (sometimes called the ultraweak operator topology) and the weak operator topology. If $\mathfrak{M}$ is any linear manifold in $\mathcal{L}(\mathcal{H})$, then $\mathfrak{M}$ inherits these two topologies. A linear functional on $\mathfrak{M}$ that is continuous in the weak* [resp., weak operator] topology will be called a *weak** [resp., *weakly*] *continuous functional.* If $\mathfrak{M}$ is closed in the weak operator topology, we will call $\mathfrak{M}$ a *weakly closed subspace.* One knows from the Hahn–Banach theorem that every weak* [resp., weakly] continuous functional on $\mathfrak{M}$ has the form $[\phi]=\phi\mid\mathfrak{M}$ where ϕ is a weak* [resp., weakly] continuous functional on $\mathcal{L}(\mathcal{H})$. In this paper we will be concerned mostly with weakly continuous functionals, and therefore we remind the reader that every such functional on $\mathcal{L}(\mathcal{H})$ is a finite sum of functionals of the form $x\otimes y$ with $x, y\in\mathcal{H}$, where

$$(x\otimes y)(A)=\langle Ax, y\rangle,\quad A\in\mathcal{L}(\mathcal{H}).$$

(Weak* continuous functionals on $\mathcal{L}(\mathcal{H})$ have the form $\sum_{n=1}^{\infty} x_n\otimes y_n$, but this fact will not be needed herein.)

Let $\mathfrak{M}$ be a linear manifold in $\mathcal{L}(\mathcal{H})$. As in [11], we will use the notation $\operatorname{Ref}(\mathfrak{M})$ for the set of all operators X in $\mathcal{L}(\mathcal{H})$ such that $Xy\in(\mathfrak{M}y)^-$ for every y in $\mathcal{H}$. The subspace $(\mathfrak{M}y)^-$ will be referred to (somewhat improperly) as the *cyclic space* for $\mathfrak{M}$ *generated* by y. The following concept of reflexivity was introduced by Loginov and Sulman in [4].

DEFINITION 1. A linear manifold $\mathfrak{M}\subset\mathcal{L}(\mathcal{H})$ is said to be *reflexive* if $\operatorname{Ref}(\mathfrak{M})=\mathfrak{M}$.

It is easy to verify that $\operatorname{Ref}(\mathfrak{M})=\operatorname{Alg\,Lat}(\mathfrak{M})$ if $\mathfrak{M}$ is an algebra containing $1_{\mathcal{H}}$, and for such algebras the above definition gives the usual one of reflexive algebras. Note, however, that $\mathfrak{M}=\{0\}$ is reflexive as a subspace but not as an algebra.

In this paper we study the relationship between the reflexivity of a linear manifold $\mathfrak{M}$ in $\mathcal{L}(\mathcal{H})$ and the structure of the weakly continuous functionals on $\mathfrak{M}$. The following definition is pertinent to the kind of structure we have in mind.

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