INTEGRAL GENERATORS IN A CERTAIN QUARTIC FIELD AND RELATED DIOPHANTINE EQUATIONS

Andrew Bremner

1. Given a subring of the ring of integers in an algebraic number field K, then an effective procedure is known for determining whether or not the ring is principally generated over $\mathbf{Z}$ (see Győry [6, Corollaire 3.3]). In the case that the ring does have principal generators, then it clearly has infinitely many, since $\mathbf{Z}[\alpha]=\mathbf{Z}[m+\alpha]$ for an arbitrary integer m. If, however, one defines two algebraic integers α, α' to be equivalent if $\alpha-\alpha' \equiv 0 \bmod \mathbf{Z}$, then Győry shows that the numbers of generators up to equivalence is finite, and effectively bounds the height of such a generator in terms of the degree of K over $\mathbf{Q}$ and the discriminant of K. Actually to determine all the generators in a given ring still seems in general a difficult question, since the bound on the height of the generators lies well beyond present computing power. Nagell [10] solves this problem in the three quartic fields corresponding to the fifth, eighth, and twelfth roots of unity. An equivalent formulation of the problem is to determine all those β in the number ring $\mathbf{Z}[\alpha]$ of index 1; or again, to determine all those β in $\mathbf{Z}[\alpha]$ satisfying discriminant $(\alpha)=$ discriminant (β). Nagell [11] in a later paper observes that in the field $\mathbf{Q}(\xi)$, $\xi^4-\xi+1=0$, then the discriminants of $\xi, \xi^2, \xi^3, \xi^4, \xi^6, \xi^7$ are all equal to 229, and notes that it is not known if the discriminant of ξ^m can equal 229 for $m>7$.

In this paper we solve this problem as a corollary to finding all the generators for the ring of integers $\mathbf{Z}[\xi]$ in $\mathbf{Q}(\xi)$. This in turn is achieved by solving in integers the Diophantine equation $G^2+6183=4H^3$; this latter involves a considerable amount of numerical detail about six particular quartic extensions of $\mathbf{Q}$. In particular, a standard algorithm for computing units had to be strengthened in order that calculations by computer could be effective. I wish to thank here the referee for appreciably improving the presentation of this paper.

2. We consider the quartic field $\mathbf{Q}(\xi)$, where $\xi^4-\xi+1=0$, and wish to determine those α in $\mathbf{Z}[\xi]$ with $\mathbf{Z}[\alpha]=\mathbf{Z}[\xi]$. Denote the conjugates of ξ by $\xi_1=\xi, \xi_2, \xi_3, \xi_4$ and similarly define α_i, $i=1,\dots,4$. Since $\operatorname{disc}(\alpha)=\operatorname{disc}(\xi)$ and $\operatorname{disc}(\alpha)=\prod_{1\le i<j\le 4}(\alpha_i-\alpha_j)^2$,

$$\prod_{1\le i<j\le 4}\left(\frac{\alpha_i-\alpha_j}{\xi_i-\xi_j}\right)=\pm 1. \tag{1}$$

Now if i,j,k,l is a permutation of $1,2,3,4$, then $\xi_i\xi_j+\xi_k\xi_l$ is a zero of the resolvent cubic equation associated to the quartic polynomial x^4-x+1, namely the equation $\Xi^3-4\Xi-1=0$. Simple Galois theory shows that

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