STABILIZING SURFACE SYMMETRIES

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This paper concerns free, orientation preserving actions of finite groups on closed, connected, oriented surfaces. Given such a group G and surface M, such an action consists by definition of an injective homomorphism $\phi: G \to \text{Diff}_+(M)$, where $\text{Diff}_+(M)$ is the group of orientation preserving diffeomorphisms of M, and $\phi(g)$ is fixed point free for all $g \in G$, other than the identity. Two such actions, ϕ_1 and ϕ_2, of G on surfaces M_1 and M_2 are called *equivalent* if there is an orientation preserving diffeomorphism $h: M_1 \to M_2$ such that $h \circ \phi_1(g) \circ h^{-1} = \phi_2(g)$ for all $g \in G$.

A general motivating question concerning such actions is, for fixed group, to determine all possible equivalence classes of actions of that group on a given surface. This has been done for cyclic groups by Nielsen [3] and for abelian and metacyclic groups by Edmonds [1; 2]. The results in those cases essentially state that two actions are equivalent if they are freely bordant. We note here that the free bordism group, $\Omega_2^{\text{free}}(G)$, is isomorphic to $H_2(G, \mathbf{Z})$. For further results including the nonorientable case see [4; 5].

As an example, the $\mathbf{Z}_5$ actions on $S^1 \times S^1$ generated by the maps $(x, y) \to (x, \omega^i y)$, where ω is a primitive fifth root of unity and i is 1 or 2, are equivalent. Finding the equivalence h is an enlightening exercise.

Given an action of G on a surface M there is a natural way to stabilize the action to an action on $M \#_k T^2$, where $k = \text{order}(G)$, and $M \#_k T^2$ denotes the connected sum of M with k copies of $T^2 = S^1 \times S^1$. Essentially we let G freely permute the added tori. This will be defined precisely in Section 1. Two actions are called *stably equivalent* if upon repeated stabilization they become equivalent. The main result of this paper is the following:

THEOREM. Let ϕ_1 *and* ϕ_2 *be free, orientation preserving actions on connected closed oriented surfaces* M_1 *and* M_2. *(*M_1 *and* M_2 *need not be homeomorphic.) Then* ϕ_1 *and* ϕ_2 *are stably equivalent if they are freely bordant.*

The converse is trivially true, as any action is bordant to its stabilization, and equivalent actions are bordant.

Whether or not stable equivalence implies equivalence for group actions is an open question. As noted earlier, such an implication holds in the case of cyclic, abelian, and metacyclic groups. In addition, it has been verified for a variety of other groups.

An outline of the paper is as follows. In Section 1 preliminary material is presented. Section 2 contains a proof of the main theorem. In the final section extensions of the main result to the unoriented or nonfree setting are summarized.

Received July 19, 1984.

The author was supported in part by a grant from the NSF.

Michigan Math. J. 32 (1985).