COMPACT COMPOSITION OPERATORS ON $H^p(B_N)$

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Introduction. Let B_N be the open unit ball in $\mathbf{C}^N$ and let $\Phi: B_N \to B_N$ be a holomorphic self-map of B_N. For f a holomorphic function on B_N, denote the composition $f \circ \Phi$ by $C_\Phi(f)$. This will again be a holomorphic function on B_N. We are concerned here with the question of when C_Φ, called the composition operator induced by Φ, will be a *bounded,* or respectively *compact,* operator on some Hardy space $H^p(B_N)$, for $0 < p < \infty$. Several authors ([1], [5]) have recently given examples to show that, in contrast to the case $N = 1$, C_Φ may indeed fail to be bounded on $H^p(B_N)$ when $N > 1$ and $p < \infty$. In Section 1 we give a necessary and sufficient condition, in terms of the measure $\sigma(\Phi^*)^{-1}$, for C_Φ to be bounded (respectively compact) on $H^p(B_N)$, and derive some consequences of this criterion.

In one variable, compact composition operators on the spaces $H^p(\mathbf{D})$ have been studied by J. Shapiro and P. Taylor in [9], where they examine the relationship between compactness of the operator C_Φ and certain geometric conditions on $\Phi(\mathbf{D})$. In particular, they show that any map Φ for which the range of Φ is contained in a region which touches the unit circle sufficiently "infrequently and sharply" will induce a compact composition operator. In Section 2 we study the question of whether there are geometric conditions on $\Phi(B_N)$ $(N > 1)$ which will guarantee that C_Φ be compact on $H^p(B_N)$. It is the existence of unbounded composition operators when $N > 1$ which makes this question much more difficult in several variables than in the case $N = 1$. Using the compactness criterion developed in Section 1, we show that any Φ with $\Phi(B_N)$ contained in a sufficiently small (depending on the dimension N) Koranyi approach region $D_\alpha(\zeta)$ will induce a compact composition operator on every $H^p(B_N)$, $p < \infty$. We give an example to show that this result is sharp in a strong sense; maps into larger Koranyi approach regions may even fail to induce bounded operators.

Finally we give an example of a map $\Phi: B_2 \to B_2$ for which C_Φ is compact on $H^p(B_2)$, but is not Hilbert-Schmidt on $H^2(B_2)$. To do this we use techniques developed in this paper to modify examples given in [9] for the case $N = 1$ of composition operators which are compact but not Hilbert-Schmidt on $H^2(\mathbf{D})$.

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1. A characterization of bounded (respectively compact) composition operators. The main goal of this section is a theorem which gives necessary and sufficient conditions for the operator C_Φ to be bounded (compact) on $H^p(B_N)$. We

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