PRIMES IN SHORT INTERVALS

Helmut Maier

1. Introduction. The distribution of primes in short intervals is an important problem in the theory of prime numbers. The following question is suggested by the prime number theorem: for which functions Φ is it true that

$$\pi(x+\Phi(x))-\pi(x)\sim\frac{\Phi(x)}{\log x}\quad(x\to\infty)? \tag{1.1}$$

Heath-Brown [4] proved that one can choose $\Phi(x)=x^{7/12-\epsilon(x)}$ ($\epsilon(x)\to 0$, as $x\to\infty$), which is a slight improvement of Huxley's result [5], $\Phi(x)=x^{7/12+\epsilon}$ ($\epsilon>0$ fixed). The Riemann hypothesis implies that one can take $\Phi(x)=x^{1/2+\epsilon}$. There is a large gap between these upper bounds and the known lower bounds of $\Phi(x)$. It follows from [9] that (1.1) is wrong if

$$\Phi(x)=\log x(\log\log x\log\log\log\log x/(\log\log\log x)^2).$$

A slight improvement is implicit in the author's paper [7].

On assumption of the Riemann hypothesis this gap can be narrowed considerably if an exceptional set of x-values is admitted. In 1943, A. Selberg [10] proved that, on assumption of the Riemann hypothesis, (1.1) is true for almost all x if $\Phi(x)/(\log x)^2\to\infty$ ($x\to\infty$). By "for almost all values of x" is meant that $x\to\infty$ through any sequence lying outside a certain *exceptional set* $\mathcal{E}$ of x-values, for which the Lebesgue measure of $\mathcal{E}\cap(0,u]$ is $o(u)$ for $u\to\infty$. It is known unconditionally that (1.1) is true for almost all x if $\Phi(x)=x^{1/6+\epsilon}$. This is implicit in the work of Huxley [5].

A natural question is whether Selberg's result is true without exceptions. The purpose of this paper is to show that *exceptions do exist* even for functions $\Phi(x)$ growing considerably faster than $(\log x)^2$.

We prove the following.

THEOREM. *Let* $\Phi(x)=(\log x)^{\lambda_0}$, $\lambda_0>1$. *Then*

$$\limsup_{x\to\infty}\frac{\pi(x+\Phi(x))-\pi(x)}{\Phi(x)/\log x}>1\quad\textit{and}\quad\liminf_{x\to\infty}\frac{\pi(x+\Phi(x))-\pi(x)}{\Phi(x)/\log x}<1.$$

For the range $1<\lambda_0<e^\gamma$ *we have even*

$$\limsup_{x\to\infty}\frac{\pi(x+\Phi(x))-\pi(x)}{\Phi(x)/\log x}\geq\frac{e^\gamma}{\lambda_0},$$

where γ *denotes Euler's constant.*

Most of the principles of the proof already appear in [7]. At some places however we need sharper estimates.

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