UNIVALENT HARMONIC MAPPINGS ONTO PARALLEL SLIT DOMAINS

W. Hengartner and G. Schober

Zu Dim Geburtstag wünsched mer
Dir Glück und Gsundheit, Freud und Ehr!
(For George Piranian on his 70th)

1. Introduction. Let D be any domain of $\bar{\mathbf{C}}$ that contains the point at infinity. It is well known that for each $c \in \mathbf{C} \setminus \{0\}$ there is a (univalent) conformal mapping ϕ_c of D onto the complement of horizontal slits and points, normalized by

$$\phi_c(z) = cz + o(1) \quad \text{as } z \to \infty.$$

Such mappings can be obtained by solving the linear extremal problem $\max \operatorname{Re}\{cb_1\}$ over all conformal mappings f of D with expansion

$$f(z) = cz + \frac{b_1}{z} + \cdots$$

near infinity.

Many authors [1, 2, 4, 5, 6, 7, 8] have generalized this result to univalent, canonical slit mappings satisfying the partial differential equation

$$f_{\bar{z}} = \mu f_z + \nu \overline{f_z} \text{ in } D, \tag{1}$$

where μ and ν satisfy the uniform ellipticity condition $\sup_D(|\mu|+|\nu|)<1$ and where D is finitely connected.

In this article D may have arbitrary connectivity, and we are interested in the equation (1) with $\mu \equiv 0$. We shall assume that ν is an anti-analytic function and $|\nu|<1$ in D, but we shall permit $|\nu|$ to approach one at the boundary. We shall obtain horizontal slit mappings which are locally quasiconformal, harmonic mappings.

2. Existence. Let a be analytic in D and satisfy $|a|<1$. Then diffeomorphic solutions of

$$f_{\bar{z}} = \bar{a}\overline{f_z} \tag{2}$$

will be locally quasiconformal in D, but the distortion as measured by the dilatation quotient $(|f_z|+|f_{\bar{z}}|)/(|f_z|-|f_{\bar{z}}|) = (1+|a|)/(1-|a|)$ may be unbounded at the boundary. In addition, since $f_{\bar{z}z} = \bar{a}\overline{f_{z\bar{z}}}$ where $|a|<1$, the mapping satisfies $f_{z\bar{z}} = 0$ and thus is harmonic. Conversely, each univalent, orientation-preserving, harmonic mapping f of D satisfies (2) for some analytic function a with $|a|<1$.

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