ON QUASICONVEX FUNCTIONS

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Dedicated to Professor George Piranian on his seventieth birthday

1. Introduction. Let $f(z)$ be a convex univalent function not assuming the value d and let $F \equiv f$ or

$$F(z) = \frac{af(z)+b}{f(z)-d} = \sum_{0}^{\infty} c_n z^n \tag{1.1}$$

be a Möbius transformation of f. We shall call such functions $F(z)$ quasiconvex and denote the class of all such functions by Q. The class Q was considered by R. R. Hall [2] who proved the following.

THEOREM A. *If $F \in Q$ then for $|z| = \rho$*

$$|F(z) - c_0| < \frac{\pi^2 |c_1|}{2} \frac{\rho}{(1-\rho)},$$

and hence $|c_n| \le A_0 |c_1|$, where A_0 is an absolute constant.

Further results were obtained by Barnard and Schober [1], by variational techniques. They denoted by $\hat{K}$ the subclass of Q for which $c_0 = 0$ and $c_1 = 1$ and by K the class of normalized convex functions and proved the following.

THEOREM B. *If $0 < r < 1$, $F \in \hat{K}$, and*

$$m(r,F) = \inf_{|z|=r} |F(z)|, \qquad M(r,F) = \sup_{|z|=r} |F(z)|,$$

then the extreme values of $m(r,F)$ and $M(r,F)$ for given r and $F \in \hat{K}$ occur when $f(z)$ maps the unit disk Δ onto a vertical strip. In particular $\max|c_2| = 1.3270$.

2. Statement of new results. Although Theorem B gives sharp results for the class $\hat{K}$, the individual bounds usually seem to be solutions of rather complicated transcendental equations. This is certainly the case for $|c_2|$. If we confine ourselves to the subclass of Q consisting of functions omitting a fixed value (e.g., zero), the bounds become more manageable and only elementary considerations such as subordination are necessary. The extremals turn out once again to be the functions considered by Barnard and Schober [1]. We have the following results.

THEOREM 1. *If $F \in Q$, and $F(z) \ne 0$ in Δ, then we have the sharp inequality*

$$|c_1| \le \frac{8}{\pi} |c_0|. \tag{2.1}$$

Further we have for $|z| = \rho$ the sharp inequalities

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