ON THE FREQUENCY OF MULTIPLE VALUES OF A MEROMORPHIC FUNCTION OF SMALL ORDER

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For George Piranian on the occasion of his alleged retirement

Introduction. We start from Nevanlinna's fundamental inequality

$$\sum_{j=1}^{q} m(r,a_j) \le \{2+o(1)\}T(r,F) - N_1(r) \quad (r\to\infty,\ r\notin \mathcal{E}) \tag{1}$$

and ask for estimates of

$$N_1(r) = N_1(r;F) = N(r,0;F') + 2N(r,\infty;F) - N(r,\infty;F'). \tag{2}$$

Here F is meromorphic and non-rational in $\mathbf{C}$, the a_j are distinct values in $\mathbf{C}\cup\{\infty\}$, and $\mathcal{E}=\mathcal{E}(F)\subset(0,\infty)$ has finite measure. The standard notations and results of value distribution theory used here are explained in the classic texts ([10], [13]). As usual we denote by

$$\delta(a,F) = \liminf_{r\to\infty} \frac{m(r,a)}{T(r,F)}$$

the Nevanlinna deficiency of a for F.

We consider

$$\Phi_1(F) = \inf_{A\in\mathcal{L}} \limsup_{\substack{r\to\infty \\ r\in A}} \frac{N_1(r)}{T(r,F)}, \tag{3}$$

where $\mathcal{L}$ is the collection of sets $A\subset(0,\infty)$ of density one (cf. [9, p. 205]), rather than the usual index of total ramification $\Phi(F) = \liminf N_1(r)/T(r,F)$, and prove the following

THEOREM 1. *If F has lower order $\mu<\frac{1}{2}$, then*

$$\Phi_1(F) \ge \cos \pi\mu. \tag{4}$$

As a direct consequence of (4), and the simple inequality $T(r,F')/T(r,F) \le 2+o(1)$ $(r\to\infty,\ r\notin\mathcal{E})$, we have the following

COROLLARY. *If F has only simple poles, then*

$$\delta(0,F') \le 1-\tfrac{1}{2}\cos\pi\mu \quad (0\le\mu<\tfrac{1}{2}). \tag{5}$$

It is not difficult to achieve $\delta(0,F')=1$ for F of any order $\mu\ge 0$, by allowing F to have poles of arbitrarily high multiplicity.

Our estimates (4) and (5) are unlikely to be sharp: the simple examples $F_\mu(z) = 1/g(z;\mu)$, where g is a Lindelöf function [13, p. 225], have

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