SMOOTHNESS OF INVERSE LAPLACE TRANSFORMS OF FUNCTIONS UNIVALENT IN A HALF-PLANE

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Dedicated to George Piranian

We denote by $\mathcal{H}$ the set of functions f univalent and analytic in the right half plane $\operatorname{Re} z > 0$ which satisfy the condition

$$\limsup_{x\to\infty} x^2|f(x)| \leqq 1.$$

This class was introduced by Hayman [9], who proved that each function $f \in \mathcal{H}$ is the Laplace transform of a function $a(t)$:

$$f(z) = \int_0^\infty a(t)e^{-tz}\,dt, \quad \operatorname{Re} z > 0.$$

The "Koebe function" for $\mathcal{H}$ is $k(z) = z^{-2}$, the corresponding inverse transform being $a(t) = t$. Hayman [9, p. 6] showed that

$$(1) \qquad \int_{-\infty}^{\infty} |f(1+iy)|\,dy \leqq \int_{-\infty}^{\infty} |k(1+iy)|\,dy = \pi, \quad f \in \mathcal{H}.$$

Set $a(t) = 0$ for $t \leqq 0$. The inverse Fourier transform of $f(1+iy)$ is $a(t)e^{-t}$. Hence $a(t) \in C(\mathbf{R})$ and

$$K_0 = \sup_{f\in\mathcal{H}} |a(1)|$$

is finite. If $f \in \mathcal{H}$ and $\lambda > 0$ then $\lambda^2 f(\lambda z) \in \mathcal{H}$ and the inverse transform of $\lambda^2 f(\lambda z)$ is $\lambda a(t/\lambda)$. We deduce that

$$|a(t)| \leqq K_0 t, \quad 0 < t < \infty, \; f \in \mathcal{H}.$$

Such "homogeneity" arguments will appear frequently in this paper.

One of the main results of [9] is the relation

$$K_0 = \lim_{n\to\infty} \frac{1}{n} \sup\{|a_n| : f \in S\},$$

where S is the usual class of functions $f(z) = z + \sum_{n=2} a_n z^n$ univalent in $|z| < 1$. Hayman's "asymptotic Bieberbach conjecture" $K_0 = 1$ remained unproved until recently, the best known estimate having been Horowitz's $K_0 \leqq 1.066$ [11]. Nehari [12], and later Bombieri [3], proved that $K_0 = 1$ implies Littlewood's conjecture $|a_n| \leqq 4n|a_0|$ for the coefficients of non-vanishing univalent functions. Conversely, Hamilton [6] showed that the truth of Littlewood's conjecture implies

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