UNIVALENT MULTIPLIERS OF THE DIRICHLET SPACE

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To George Piranian with fondness and respect

Let G be a connected open set in the complex plane and fix a distinguished point $z_0 \in G$. The Dirichlet space $D(G)$ is the Hilbert space of analytic functions g on G such that $g(z_0)=0$ and

$$\|g\|_{D(G)}^2 = \int_G |g'|^2 \, dA < \infty,$$

where dA denotes the usual area measure. The Dirichlet norm squared of g is just the area of the image of G under g, counting multiplicity. The condition $g(z_0)=0$ insures that no nonzero function has norm zero. Changing the point z_0 gives a space which is obtained from the original by subtracting a suitable constant from each function. An analytic function ϕ on G is called a multiplier of $D(G)$ if $\phi D(G) \subset D(G)$.

The Bergman space $B(G)$ is the Hilbert space of analytic functions g on G such that

$$\|g\|_{B(G)}^2 = \int_G |g|^2 \, dA < \infty.$$

For the special case of the open unit disk, which we denote by U, the Dirichlet space (with the distinguished point equal to zero) and the Bergman space can be described in terms of Taylor coefficients; namely,

$$\|g\|_{D(U)}^2 = \pi \sum n|a_n|^2,$$
$$\|g\|_{B(U)}^2 = \pi \sum |a_n|^2/(n+1),$$

where $g(z) = \sum a_n z^n$.

From the Taylor coefficient formulas for the norms in $D(U)$ and $B(U)$, it is clear that $D(U)$ is contained in $B(U)$. In this paper we consider the question of when the Dirichlet space $D(G)$ is contained in the Bergman space $B(G)$. Our results deal primarily with the case where G is bounded and simply connected. We show that if G is bounded and starlike, then $D(G)$ is contained in $B(G)$ (Theorem 3). Theorem 1 shows that a Riemann map ϕ of the unit disk U onto G is a multiplier of $D(U)$ precisely when $D(G) \subset B(G)$. Corollary 7 shows that if ϕ' is in H^p for some $p>1$, then $D(G) \subset B(G)$. We show that this conclusion may fail when $p=1$. In Theorem 10 we construct a Jordan region G with a rectifiable boundary such that $D(G)$ is not contained in $B(G)$. For this region G, the function z is not a multiplier of $D(G)$. Theorem 11 identifies the essential spectrum of multipliers on $D(G)$.

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