ON CERTAIN ANALYTIC (NEVANLINNA) FUNCTIONS

Maxwell O. Reade and Pavel G. Todorov

1. Introduction. We study the classes N_1 and N_2 of all analytic functions having the representations

$$f(z)=\int_{-1}^{1}\frac{d\mu(t)}{z-t}\quad\text{and}\quad \phi(z)=\int_{-1}^{1}\frac{z\,d\mu(t)}{1-tz},\tag{1}$$

where μ is a probability measure. These classes have been the subject of some interesting research during the recent past. Thale [5] showed that the maximal domain of univalence of $N_1(N_2)$ is the open set $|z|>1(|z|<1)$. In two recent notes ([3], [4]) we found the radii of starlikeness and convexity, of order alpha, of N_1 and N_2. We also proved that for each $\phi\in N_2$, $\phi(z)$ and $z\phi'(z)$ are typically-real for $|z|<1$. On the other hand, Goluzin [2] found sharp bounds on the modulus and on the argument of the set TR of all functions $g(z)=z+\cdots$ that are typically-real for $|z|<1$. But Goluzin's extremal functions do *not* belong to our class N_2. Hence it is reasonable for us to try to obtain sharp bounds on Goluzin's functionals $|\phi(z)|$, $\arg\phi(z)$ for $\phi\in N_2$. We do just that plus more. We also find sharp bounds on $|\mathrm{Im}\,\phi(z)|$, $|\phi'(z)|$ and $\arg\phi'(z)$, for $\phi\in N_2$, $0\le|z|<1$.

2. Bounds on $|\phi(z)|$ and $|\phi'(z)|$. The kernels

$$l(z,t)\equiv\frac{z}{1-tz},\quad k(z,t)\equiv\frac{1}{z-t}\tag{2}$$

play a leading role, as we shall see.

THEOREM 1. *For each z, $|z|<1$, and for each $\phi\in N_2$, the following inequalities hold:*

$$|\phi(z)|\le\left|\frac{z}{1\pm z}\right|,\quad |z\pm\tfrac{1}{2}|\le\tfrac{1}{2},\tag{3}$$

$$|\phi(z)|\le\frac{1}{|\mathrm{Im}(1/z)|},\quad |z\pm\tfrac{1}{2}|\ge\tfrac{1}{2},\tag{4}$$

where for $z\ne0$, equality in (3) *holds only for the appropriate function $\phi(z)\equiv l(z,\pm1)$ and equality in* (4) *holds only for $\phi(z)\equiv l(z,t)$, with $t=\mathrm{Re}(1/z)$.*

Proof. We suppose $z\ne0$. The integral (1) yields

$$|\phi(z)|\le\int_{-1}^{1}|l(z,t)|\,d\mu(t).$$

Hence we study $|l(z,t)|$, $-1\le t\le1$. Let

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