TRANSCENDENTAL TRANSCENDENCE OF SOLUTIONS OF SCHRÖDER'S EQUATION ASSOCIATED WITH FINITE BLASCHKE PRODUCTS

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Dedicated to George Piranian

1. Introduction and statement of results. Let S be a finite Blaschke product with $S(0)=0$ and $S'(0)=s$, $0<|s|<1$. Let $\phi \not\equiv 0$ be a function which is meromorphic in the unit disk D and satisfies Schröder's functional equation there:

$$\phi(S(z))=s\phi(z). \tag{S}$$

Kuczma ([7], [8]) has a survey of the extensive literature on this equation. More recent work may be found in [5]. Fatou [6] showed that if a certain countable collection of disks is removed from D, then $|\phi(z)|$ approaches infinity as z approaches the unit circle C through what remains of D. This remainder contains circles $|z|=r$ with r arbitrarily close to 1; ϕ is what is now called a strongly annular function. The situation with ϕ is reminiscent of the meromorphic Tsuji function constructed by Collingwood and Piranian [4], and of infinite product annular functions in [1], [2], and [3].

A function ϕ meromorphic in D is said to be a *differentially algebraic* function if it satisfies an algebraic differential equation (ADE)

$$P(z, w(z), w'(z), \dots, w^{(n)}(z))=0,$$

where $P(z, w_0, w_1, \dots, w_n)$ is a polynomial in all its variables. If ϕ satisfies no nontrivial ADE, then ϕ is *transcendentally transcendental.* See [11].

In a private communication, L. A. Rubel asked whether an annular function can be differentially algebraic. The author knows of no such example, and undertook the present research in an attempt to settle the question. The main result of this paper is that every nontrivial solution of (S) is transcendentally transcendental.

THEOREM. *Let S be a finite Blaschke product with* $S(0)=0$, $S'(0)=s$, $0<|s|<1$. *Let ϕ be a function meromorphic in D and not identically* 0, *satisfying Schröder's equation*

$$\phi(S(z))=s\phi(z), \quad (z\in D). \tag{S}$$

Then ϕ satisfies no algebraic differential equation.

Two simple examples will illustrate the method of proof.

Received May 4, 1984. Revision received September 20, 1984.
Michigan Math. J. 32 (1985).